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Math 32A **Midterm 2** Name Yaacov Tarko

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Section C

Write everything that you want graded on these pages and clearly indicate your answers. If you need more space, use the back of these pages and clearly indicate where the continuation may be found. Write as legibly as possible since I will ignore anything I cannot read (or find). You must show your work to obtain full credit. Points may be deducted if you do not justify your final answer. No aids such as calculators, notes, and textbooks are allowed.

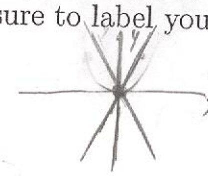
1. (25 pts) Consider the function $f(x, y) = \frac{y^2}{4} - x^2$.

   (a) (12 pts) Sketch the level curves $f(x, y) = 0$, $f(x, y) = 4$, and $f(x, y) = -4$, indicating which one is which. Make sure to label your axes.

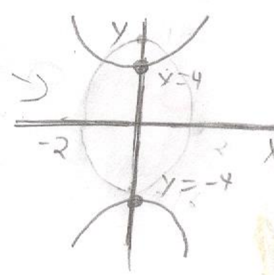
$\frac{y^2}{4} - x^2 = 0 \rightarrow$

$x^2 = \frac{y^2}{4}$

$y = \pm 2x$

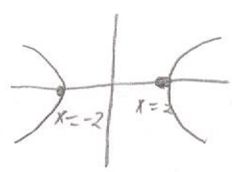


$\frac{y^2}{4} - x^2 = 4 \rightarrow$



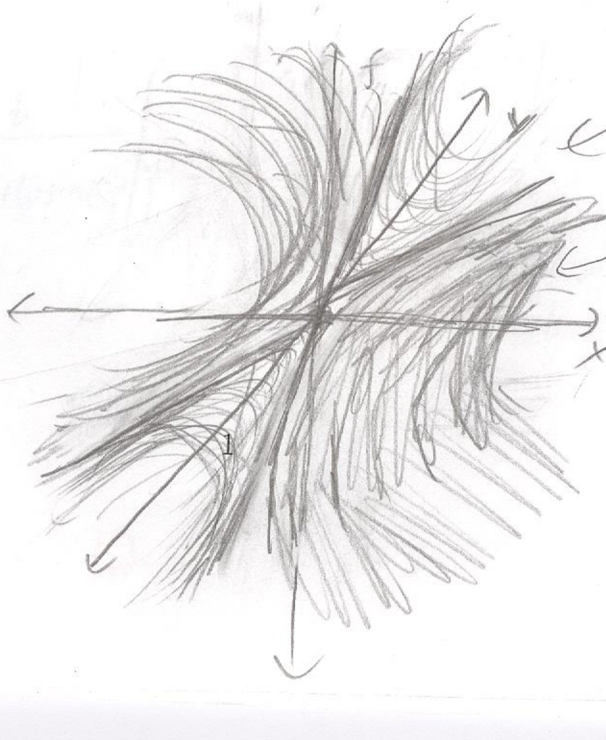
✓

$\frac{y^2}{4} - x^2 = -4$



   (b) (13 pts) Sketch the graph of the function $f(x, y)$. Make your drawing large and clear and label your axes.

-10



above f=0 hyperbol

below f=0

15

2. (10 pts each) Find the limit, if it exists, or show that the limit does not exist. You must justify your answers.

(a) $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$.

$\to\infty$

$\lim_{(x,y)\to(0,0)}$ along $(x=y^2)$ is $\lim_{y\to 0}\frac{y^3}{y^4+y^2} = \lim_{y\to 0} \frac{1}{y^{3}+\frac{1}{y}} \to 0 = \lim_{y\to 0} y = 0$

$\lim_{(x,y)\to(0,0)}$ along $(x=y)$ is $\lim_{y\to 0}\frac{y^2}{2y^2} = \frac{1}{2}$

OK 10

So the limit does not exist

(b) $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$. $= \lim_{r\to 0} \frac{r\sin\theta \cdot r\cos\theta}{\sqrt{r^2(\sin^2\theta+\cos^2\theta}} = \lim_{r\to 0}\frac{r^2\sin\theta\cos\theta}{r} =$

$\lim_{r\to 0} r\underbrace{\sin\theta\cos\theta}_{\text{always between } \pm 1} = 0$ 10 ✓

3. (15 pts) Find the curvature of the parametrization $\mathbf{r}(t) = (a\sin t, b\cos t)$ of the ellipse $(\frac{x}{a})^2 + (\frac{y}{b})^2 = 1$.

$\text{curvature} = \frac{|r' \times r''|}{|r'|^3}$

$|r'| = \sqrt{(a\cos t)^2+(b\sin t)^2}$.

$r' = \langle a\cos t, -b\sin t\rangle$

$r'' = \langle -a\sin t, -b\cos t\rangle$

$|r' \times r''| = \begin{vmatrix} i & j & k \\ a\cos & -b\sin & 0 \\ -a\sin & -b\cos & 0 \end{vmatrix} =$

$\text{curvature} = \frac{|-ab|}{((a\cos t)^2+(b\sin t)^2)^{3/2}}$

$\langle 0,0,-ab\cos^2(t)-ab\sin^2(t)\rangle =$

$\langle 0,0,-ab(\cos^2 t+\sin^2 t)\rangle = -ab$

12

$\frac{32}{30}$

4. (20 pts) Consider the function $f(x, y, z) = \ln(x^2 + y^2 + 2z^2)$.

(a) (5 pts) Compute the gradient of f.

$$\nabla f = \left(\frac{2x}{x^2+y^2+2z^2}, \frac{2y}{x^2+y^2+2z^2}, \frac{4z}{x^2+y^2+2z^2} \right)$$

✓

(b) (5 pts) Find the rate of change of f at $(1, 0, 2)$ in the direction of $\mathbf{u} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$.

$$\nabla f(a,b) \cdot u = \left(\frac{2}{1+0+2\cdot 2^2}, 0, \frac{4\cdot 2}{1+0+2\cdot 2^2} \right) \cdot \vec{u} =$$

$$✓ \left(\frac{2}{9}, 0, \frac{8}{9} \right) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right) = \frac{2}{9\sqrt{2}}$$

(c) (4 pts) In which direction is f *decreasing* the fastest, at the point $(1, 0, 2)$?

∇f at $(1,0,2)$ is $\left(\frac{2\cdot 1}{1+0+8}, 0, \frac{8}{1+0+8} \right) = \left(\frac{2}{9}, 0, \frac{8}{9} \right)$

decreases the fastest at $-\nabla f = -\left(\frac{2}{9}, 0, \frac{8}{9} \right)$

✓

(d) (6 pts) Give an equation for the tangent plane to the level surface $f(x, y, z) = \ln 9$ at the point $(1, 0, 2)$.

$$D_f = \nabla f(r) \cdot (\vec{p} - \vec{p}_0)$$

0

If $f(x,y,z) = c$,

A normal vector at $\vec{p}$ is $\nabla f(\vec{p})$

(here $\left(\frac{2}{9}, 0, \frac{8}{9} \right)$

$$n \cdot (p - p_0)$$

$$\frac{2}{9}(x-1) + 0(y-0) + \frac{8}{9}(z-2) = 0$$

14

5. (20 pts) A bullet is fired at an angle of 45° at a tower located 100 m away, with initial speed 100 m/s. Find the height at which the bullet hits the tower. *Assume that the acceleration due to gravity is 10 m/s^2.*

20

$|v_0| = 100$

$V_0 = k\sin 45, \ k\cos 45 =$

$\left(\frac{\sqrt{2}}{2}k, \frac{\sqrt{2}}{2}k\right) = (c, c)$

$\sqrt{(c)^2+(c)^2} = \sqrt{2}\cdot\sqrt{c^2} = 100$

$c = \frac{100}{\sqrt{2}}$

$V_0 = \left(\frac{100}{\sqrt{2}}, \frac{100}{\sqrt{2}}\right)$

$V_{(t)} = \left(\frac{100}{\sqrt{2}}, \frac{100}{\sqrt{2}} - 10t\right)$

$P_{(t)} = \left(\frac{100}{\sqrt{2}}t, \frac{100}{\sqrt{2}}t - 5t^2\right)$ ✓

$\frac{100}{\sqrt{2}}t = 100$ ✓

$t = \sqrt{2}$ ✓

$\frac{100}{\sqrt{2}}\cdot\sqrt{2} - 5\cdot\sqrt{2}^2 = 100 - 10 = \boxed{90\text{ m}}$ ✓

20