1. (a) (3 points) Let $f(x, y) = x^2 + y^2$. Compute the partial derivative $f_{xx}$.

$$f_{xx} = \frac{d}{dx} 2x = 2$$

(b) (5 points) Let $f(u, v, w, x, y, z) = u^2/v + vxyz + \sin(xwv)$. Compute the partial derivative $f_{xvz}$.

$$f_{xvz} = \frac{d}{dx} \frac{d}{dv} vxy = \frac{d}{dx} xy = y$$

(c) (5 points) Compute the following limit:

$$\lim_{(x,y)\to(0,2)} (1+x)^{y/x}.$$

$$L = \lim_{(x,y)\to(0,2)} (1+x)^{\frac{y}{x}}$$

$$\ln L = \ln \lim_{(x,y)\to(0,2)} (1+x)^{\frac{y}{x}}$$

$$= \lim_{(x,y)\to(0,2)} \frac{y}{x} \ln(1+x)$$

$$= \lim_{x\to 0} \frac{2\ln(1+x)}{x}$$

L'Hopital Rule

$$= \lim_{x\to 0} \frac{2\frac{1}{1+x}}{1}$$

$$= \lim_{x\to 0} \frac{2}{1+x} = 2.$$

$$\therefore L = e^2$$

$$\therefore \lim_{(x,y)\to(0,2)} (1+x)^{\frac{y}{x}} = e^2$$

4

2. (10 points) Let $f(x, y) = x^2y^3$. Compute the gradient $\nabla f(x, y)$. Then, find the tangent plane to the surface $z = f(x, y)$ at the point $(a, b) = (2, 3)$.

10

$$\nabla f(x,y) = (2xy^3,\ 3y^2x^2)$$

$$L = f(a,b) + \big((x,y)-(a,b)\big)\cdot \nabla f(a,b)$$

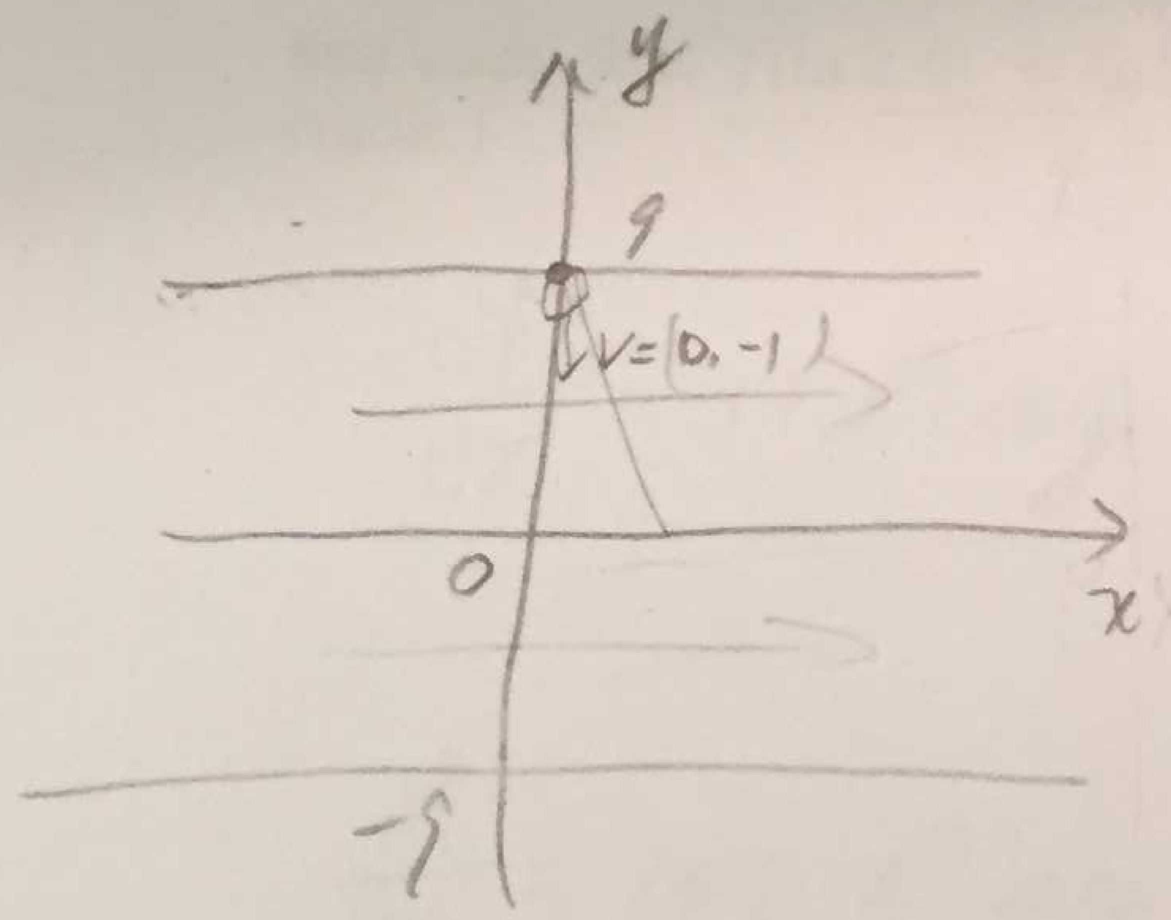
$$= f(2,3) + (x-2,\ y-3)\cdot(27\times 4,\ 27\times 4)$$

27
4
108

$$= 108 + 108x - 2\cdot 108 + 108y - 3\cdot 108$$

$$= 108x + 108y - 432$$

3. (10 points) Suppose your initial position in the plane is $(0, 9)$. Between the lines $y = 9$ and $y = -9$ is a river. The river's speed at the point $(x, y)$ is $81 - y^2$, where the river runs in the direction of the positive $x$-axis. Suppose you are in a boat which begins with a constant speed of 1, in the negative $y$-direction. (So, the velocity of the boat in the $y$-direction will always be $-1$.) What will be your position when you reach the bottom of the river? That is, what is your position when you reach the line $y = -9$?




$V_{boat}(x,y) = (0,-1)$

$V_{river}(x,y) = (81-y^2, 0)$

$V_{total}(x,y) = V_{boat} + V_{river} = (81-y^2, -1)$

- time $= t$ $\quad t = 18$

$0 < t \leq 9 \quad t = y \quad \int_0^9 81 - y^2 \, dy = 81y - \frac{y^3}{3}$

$= 81 \cdot 9 - \frac{9^3}{3}$

$=$

horizontal distance traveled : $d = \int_{-9}^{9} 81 - y^2 \, dy$

$d = 81y - \frac{y^3}{3} \Big]_{-9}^{9}$

$= 81 \cdot (9) - \frac{9^3}{3} - \left[(-9)(81) + \frac{9^3}{3}\right]$

$= 729 + 729 = 1458$

$\therefore$ position is $(1458, -9)$

6



$81 \times 9 = 729$ $\quad 1944 - 1458 = 486$ $\quad 18 \times 81 = 1458$ $\quad 18 \times 18 = 324$

4. (10 points) Find a function $f(x, y)$ such that

$$\frac{\partial f}{\partial x} = 1 + e^x \cos y,$$

$$\frac{\partial f}{\partial y} = 14y - e^x \sin y,$$

and such that $f(\ln 2, 0) = \ln 2$. (As usual, you must show your work to receive full credit.)

$$\frac{\partial f}{\partial x} = 1 + e^x \cos y \qquad \int \frac{\partial f}{\partial x} = \int (1 + e^x \cos y)\, dx$$

$$= x + e^x \cos y + C$$

$$\frac{\partial f}{\partial y} = 14y - e^x \sin y \qquad \int \frac{\partial f}{\partial y} = \int 14y - e^x \sin y \, dy$$

$$= 7y^2 + e^x \cos y + c$$

$$\therefore f(x,y) = x + 7y^2 + e^x \cos y + c$$

$$f(\ln 2, 0) = \ln 2 + e^{\ln 2} \cdot 1 + c$$

$$\ln 2 = \ln 2 + 2 + c$$

$$c = -2$$

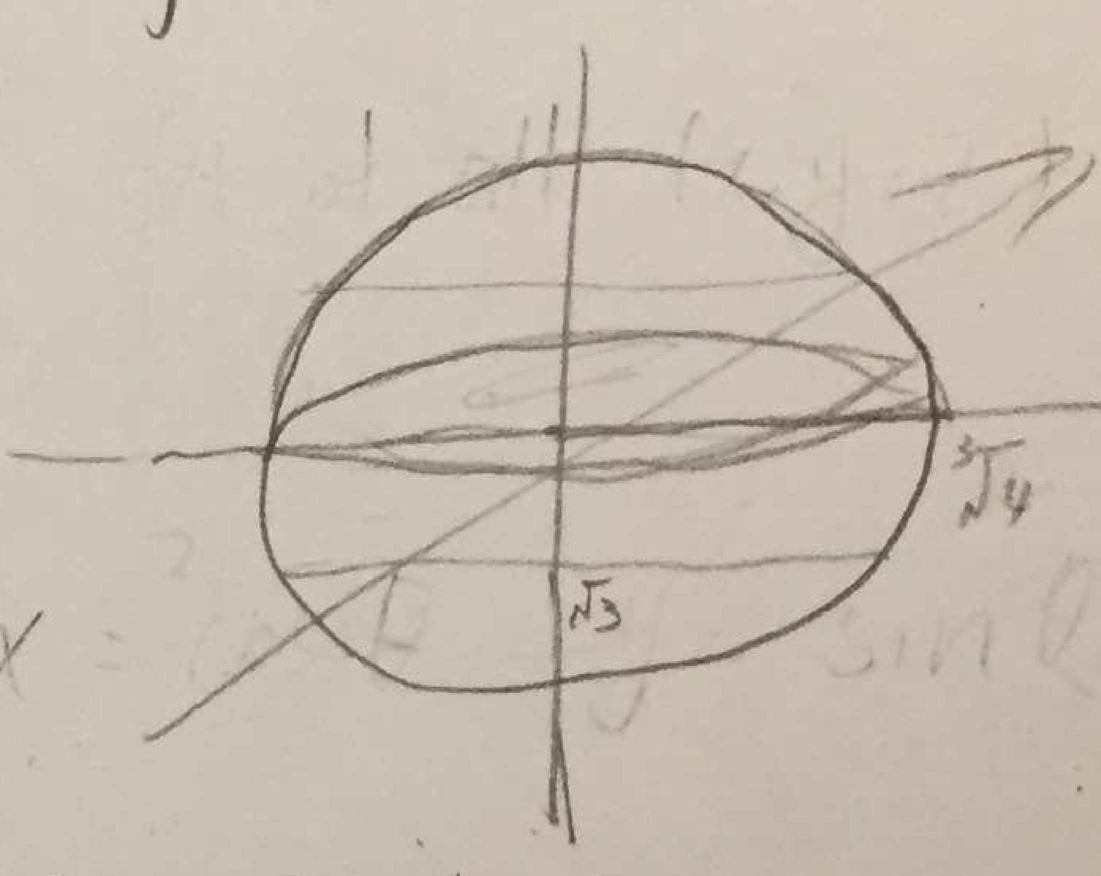
$$\therefore f(x,y) = 7y^2 + x + e^x \cos y - 2$$

5. (15 points) Let $D$ be the solid region in Euclidean space $\mathbf{R}^3$ defined as the set of all $(x, y, z)$ such that $x^2 + y^2 + z^2 \leq 4$, $x^2 + y^2 - 3z^2 \geq 0$ and $x^2 + y^2 \geq 1$. Note that $D$ is a solid region, so its boundary is a surface. Let $E$ denote the boundary surface of $D$. (If the solid region $D$ were dipped in paint, then the boundary of $D$ is the outer part of $D$ that is covered in paint.)

Let $B$ be the region in Euclidean space $\mathbf{R}^3$ defined as the set of all $(x, y, z)$ such that $y = x$, $x \geq 0$ and $y \geq 0$. Then $E$ and $B$ are surfaces.

Parametrize the intersection of $E$ and $B$. (Make sure to parametrize the entire intersection. You **MUST** specify the domain of your parameter for any parametrization you give.) (Any parametrization that you write **MUST** use $z$ as a parameter. That is, any parametrization you write must be of the form $r(z) = (x(z), y(z), z)$, where $x$ and $y$ are both functions of $z$.)

$$D: \begin{cases} x^2+y^2+z^2 \leq 4 \\ x^2+y^2-3z^2 \geq 0 \\ x^2+y^2 \geq 1 \end{cases} \Rightarrow \begin{cases} z^2 \leq 3 \\ 3z^2 \leq x^2+y^2 \leq 4-z^2 \end{cases}$$




let $x$

$$E = \begin{cases} x^2+y^2 = 1 \\ x^2+y^2 \geq 3z^2 \\ x^2+y^2 \leq 4-z^2 \end{cases}$$

X