Problem 5. (4)

Assume that at a certain moment $t = t_0$, a moving particle has velocity $\mathbf{v} =< 2, -1, 1 >$ and acceleration $\mathbf{a} =< 1, 2, -3 >$.

(i) Decide if the particle is speeding up at the moment $t = t_0$.

(ii) What is the curvature of the path of the particle at the moment $t = t_0$?

(i) $(f(\vec{c}(t))' = \nabla f\ \vec{c}_t \cdot \vec{c}\,'(t)$ $\nabla f(\vec{c}(t)) =$ speed of particle. $\vec{c}(t) =$ path of the particle

$= v_{t_0} \cdot a_{t_0}$

$= \langle 2,-1,1\rangle \cdot \langle 1,2,-3\rangle$.

$= 2-2-3 = \boxed{-3}$

$\boxed{\text{therefore the particle is slowing down}}$ ✓

(ii) curvature $k(t) = \dfrac{\|r' \times r''\|}{\|r'\|^3}$.

$r'(t) = v(t)$.

$r''(t) = a(t)$.

$\therefore k(t_0) = \dfrac{\|\langle 2,-1,1\rangle \times \langle 1,2,-3\rangle\|}{\|\langle 2,-1,1\rangle\|^3}$

$r' \times r'' = \begin{vmatrix} i & j & k \\ 2 & -1 & 1 \\ 1 & 2 & -3 \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 2 & -3 \end{vmatrix} i - \begin{vmatrix} 2 & 1 \\ 1 & -3 \end{vmatrix} j + \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} k$

$= i + 7j + 5k$

$= \langle 1, 7, 5\rangle$

$\|r' \times r''\| = \sqrt{1+49+25} = \sqrt{75} = 5\sqrt{3}$

$\|r'\| = \sqrt{4+1+1} = \sqrt{6}$

$\|r'\|^3 = (\sqrt{6})^3 = 6\sqrt{6}$.

$\boxed{\therefore k(t_0) = \dfrac{5\sqrt{3}}{6\sqrt{6}} = \dfrac{5}{6\sqrt{2}}}$

+4

1	
2	
3	
4	3
5	
T	

MATH 32A Midterm II , Winter 2013

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Instruction: Justify all your answers.

Problem 1. (4)

A bullet is fired from ground with angle of elevation α and initial speed $v_0 = 100m/s$. The only external force is due to gravity. You may assume that the initial position $\mathbf{r}(0) = (0, 0)$.

(i)Find the position vector $\mathbf{r}(t)$.

(ii)The range of the bullet is defined to be the distance between the point where the bullet is fired and the point where it hits the ground. Find the angle α which maximizes the range (justify your answer). What is the maximal range?

Note: The gravity constant $g = 9.8(m/s^2)$.

i) $a(t) = \langle 0, -9.8 \rangle$

$v(t) = \int a(t) + V_0$

$= \langle 0, -9.8t \rangle + \langle 100\cos\alpha, 100\sin\alpha \rangle$

$= \langle 100\cos\alpha, -9.8t + 100\sin\alpha \rangle$

$r(t) = \int v(t) + r_0$

$= \boxed{\langle 100t\cos\alpha, -4.9t^2 + 100t\sin\alpha \rangle}$

ii) $-4.9t^2 + 100t\sin\alpha = 0$

$t(-4.9t + 100\sin\alpha) = 0$

$4.9t = 100\sin\alpha$

$t = \dfrac{100\sin\alpha}{4.9}$

+4

$$t = \frac{100 \sin\alpha}{4.9}$$

$$r_x(\alpha) = 100\left(\frac{100\sin\alpha}{4.9}\right)\cos\alpha.$$

$$= \frac{10000}{4.9}\sin\alpha\cos\alpha.$$

$$= \frac{10000}{9.8}\sin 2\alpha.$$

optimization to get maximum.

$$r_x'(\alpha) = \frac{20000}{9.8}\cos 2\alpha = 0.$$

$$\cos 2\alpha = 0.$$

$$\boxed{\therefore \alpha = \frac{\pi}{4} = 45^\circ}$$

maximum range.

$$\frac{10000}{9.8}\sin\frac{\pi}{2} = \frac{10000}{9.8} = \frac{100000}{98}$$

$$\boxed{\frac{100000}{98}\,m}$$

$$\frac{a(b-2)^2}{a^2+(b-2)^2}+b.$$

$ab^2-4ab+4a$

Problem 2. (4)

(i) Find $\lim_{(x,y)\mapsto(a,b)} \frac{x(y-2)^2}{x^2+(y-2)^2}+y$.

Note: Your answer should be in term of a and b.

(ii) Function f is defined to be

$$f(x,y)=\begin{cases} \frac{x(y-2)^2}{x^2+(y-2)^2}+y, & \textit{when } x\neq 0 \\ 0, & \textit{when } x=0 \end{cases}$$

Decide where f is continuous and where it is not.

(i) if $a\neq 0$, $b\neq 2$,

$$\lim_{(x,y)\to(a,b)} \frac{x(y-2)^2}{x^2+(y-2)^2}+y$$

$$=\frac{a(b-2)^2}{a^2+(b-2)^2}+b.$$

if $a=0$, $b\neq 2$.

$$\lim_{(x,y)\to(a,b)} f(x,y)=b.$$

if $a\neq 0$, $b=2$

$$\lim_{(x,y)\to(a,b)} f(x,y)=2$$

$+2.$

if $a=0$, $b=2$

let $y-2=z$

$$\lim_{(x,z)\to(0,0)} \frac{xz^2}{x^2+z^2}+(z+2)=\lim_{(x,z)\to(0,0)}\frac{xz^2}{x^2+z^2}+\lim_{z\to 0} z+2$$

$$=\lim_{(x,z)\to(0,0)}\frac{xz^2}{x^2+z^2}+2$$

let $x=r\cos\theta$
$z=r\sin\theta$

$$\lim_{r\to 0}\frac{r^3\cos\theta\sin^2\theta}{r^2\cos^2\theta+r^2\sin^2\theta}+2=\lim_{r\to 0}\frac{r^3\cos\theta\sin^2\theta}{r^2}+2.$$

$$=\lim_{r\to 0} r\cos\theta\sin^2\theta+2$$

$-1 \leq \cos\theta \sin^2\theta \leq 1$.

$-|r| \leq r\cos\theta\sin^2\theta \leq |r|$

$\lim_{r \to 0} -|r| = 0$

$\lim_{r \to 0} |r| = 0$.

therefore, by squezze theorem,

$\lim_{r \to 0} r\cos\theta \sin^2\theta = 0$

$$\boxed{\text{therefore, } \lim_{(x,y)\to(0,2)} \frac{x(y-2)^2}{x^2+(y-2)^2} = 2}$$

ii) in order to make f continuous, at a point (a,b).

i) $f(a,b)$ exists

ii) $\lim_{(x,y)\to(a,b)} f(x,y)$ exists

iii) $f(a,b) = \lim_{(x,y)\to(a,b)} f(x,y)$.

at all points except for $(0,2)$,

$f(a,b) = \lim_{(x,y)\to(a,b)} f(x,y) = \frac{a(b-2)^2}{a^2+(b-2)^2} + b$ $= f(a,b)$ when $a \neq 0$.

+1

therefore continuous at all point (a,b) ~~except for $(0,2)$~~

where $a \neq 0$.

but at $(0,2)$,

$\lim_{(x,y)\to(0,2)} f(x,y) = 2$

Now check continuity at point $(0,b)$ where $b \neq 2$.

$f(x,y) = 0$ $\quad 2 \neq 0$. $\quad x=0$

therefore it is not continuous on $(0,2)$

+0.5

Problem 3. (4)

Let $f = e^{\cos x} \sin y - 1$.

(i) Find the tangent plane of the graph of the function $z = f(x, y)$ at the point $x = 0, y = 0$.

(ii) Estimate the value $f(-0.02, 0.01)$.

Note: The value e is about 2.78.

(i) $z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$= f(0,0) + f_x(0,0)x + f_y(0,0)y$

$f(0,0) = -1$

$f_x = -e^{\cos x} \sin x \sin y$

$f_x(0,0) = 0$

$f_y = e^{\cos x} \cos y$

$f_y(0,0) = e$

$\therefore z = -1 + ey$

$\boxed{z = -1 + 2.78y}$

$$\begin{array}{r} 1 \\ 0.0278 \\ \hline 0.9722 \end{array}$$

(ii) $f(-0.02, 0.01) = f(0,0) + f_x(0,0)(-0.02) + f_y(0,0)(0.01)$

$= -1 + 2.78(0.01)$

$= 0.0278 - 1$

$= \boxed{-0.9722}$

+4

$\nabla f \cdot v$

Problem 4. (4)

Let $f = x^2 + y^4 + z^6$.

(i) Find $\nabla f(x,y,z)$ and $\nabla f(3,2,1)$.

(ii) Find the directional derivative $D_{\mathbf{u}} f(x,y,z)$ and $D_{\mathbf{u}} f(3,2,1)$, where $\mathbf{u}$ is the unit vector in the direction of $< 1, 0, 1 >$.

(iii) Find the equation for the tangent plane of the level surface S at point $(3,2,1)$, where S is give by the equation $x^2 + y^4 + z^6 = 26$.

(i) $\nabla f(x,y,z) = \langle f_x, f_y, f_z \rangle = \boxed{\langle 2x, 4y^3, 6z^5 \rangle}$

$\boxed{\nabla f(3,2,1) = \langle 6, 32, 6 \rangle}$

+1

(ii) $D_u f(x,y,z) = \langle 2x, 4y^3, 6z^5 \rangle \cdot \dfrac{\langle 1,0,1 \rangle}{\|\langle 1,0,1 \rangle\|}$

$= \boxed{(2x + 6z^5)} / \sqrt{2}$

$D_u f(3,2,1) = \langle 6,32,6 \rangle \cdot \langle 1,0,1 \rangle$

$= \boxed{(12)} / \sqrt{2}.$ $\cancel{\langle 6,0,6 \rangle}$

+~~1/2~~ 1.5

18
64
82
6
88

(iii) $\nabla f_p \cdot \langle x-a, y-b, z-c \rangle$

$0 = \langle 6,32,6 \rangle \cdot \langle x-3, y-2, z-1 \rangle$

$0 = 6x - 18 + 32y - 64 + 6z - 6$

$\boxed{6x + 32y + 6z = 88}$

+1