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MATH 32A Midterm II, Fall 2017

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Instruction: Justify all your answers. No Points Will Be Given Without Sufficient Reasoning/Calculations.

Problem 1. (4)

(i) Let $\mathbf{r}(t) = (bt, c\sin t, c\cos t)$ be the position vector of a moving particle, where b and c are positive constants. Find the velocity $\mathbf{v}$ and acceleration $\mathbf{a}$.

(ii)Find tangential and normal components of $\mathbf{a}$ in (i).

(iii) Let C be a curve given by $\mathbf{r}(t) = (bt, c\sin t, c\cos t)$ in (i) above. Using the results you obtained in (i) and (ii) above, find the curvature of C.

i) $$\boxed{\begin{aligned}\vec{v}(t) &= \vec{r}\,'(t) = \langle b, c\cos t, -c\sin t\rangle \\ \vec{a}(t) &= \vec{v}\,'(t) = \langle 0, -c\sin t, -c\cos t\rangle\end{aligned}}$$

ii) $\vec{a} = a_T\vec{T} + a_N\vec{N}$ $\quad a_T = v'$ $\quad a_N = KV^2$

$$v(t) = \|\vec{v}(t)\| = \sqrt{b^2 + c^2\cos^2 t + c^2\sin^2 t}$$
$$= \sqrt{b^2 + c^2(\cancel{\cos^2 t + \sin^2 t})} = \sqrt{b^2+c^2}$$

$a_T = v'(t) = 0$ (v is const)

$$\boxed{a_T = 0}$$

$$\vec{a} = 0 + a_N\vec{N}$$
$$\vec{a} = a_N\vec{N}$$
$$\|\vec{a}\| = a_N = \sqrt{0 + c^2\sin^2 t + c^2\cos^2 t}$$
$$= \sqrt{c^2(\cancel{\sin^2 t + \cos^2 t})}$$
$$\boxed{a_N = c}$$

iii) $a_N = kv^2$

$$K = \frac{a_N}{v^2} = \frac{c}{\sqrt{b^2+c^2}^{\,2}} = \boxed{\frac{c}{b^2+c^2}}$$

Problem 2. (4)

A bullet is fired from ground at an angle $\alpha = \pi/4$ with initial speed v_0.

(i)Find the position vector $\mathbf{r}(t)$, assuming that the initial position $\mathbf{r}(0) = (0,0)$.

(ii)Find the initial speed v_0 if the bullet hits the point $p = (200, 100)$.

(iii)Find all points $p = (x_0, y_0)$ that can be hit by the bullet, assuming that you can adjust the initial speed v_0 as you like.

Note : the gravity constant $g = 9.8m/s^2$ and the square root of g is (about) **3.1**.

$\vec{v}(0) = \langle \cos\alpha, \sin\alpha \rangle v_0$

i) $\vec{a} = \langle 0, -g \rangle \quad \vec{v}(t) = \vec{v}(0) + \int_0^t \vec{a}(u)\,du$

$= \langle v_0\cos\alpha, v_0\sin\alpha \rangle + \int_0^t \langle 0, -g \rangle\,du$

$= \langle v_0\cos\alpha, v_0\sin\alpha \rangle + \langle 0, -gt - 0 \rangle$

$\vec{v}(t) = \langle v_0\cos\alpha, v_0\sin\alpha - gt \rangle$

$\vec{r}(t) = \vec{r}(0) + \int_0^t \vec{v}(u)\,du$

$= \langle 0,0 \rangle + \langle v_0\cos\alpha\, t, v_0\sin\alpha\, t - \frac{gt^2}{2} \rangle$

$\boxed{\vec{r}(t) = \langle \frac{v_0\sqrt{2}}{2}t, \frac{v_0\sqrt{2}}{2}t - \frac{gt^2}{2} \rangle}$ 1

ii) $p = (200, 100)$

$\frac{v_0 t}{\sqrt{2}} = \frac{v_0\sqrt{2}}{2}t = 200 \qquad \frac{v_0\sqrt{2}}{2}t - \frac{gt^2}{2} = 100$

$t = \frac{200\sqrt{2}}{v_0} \qquad \frac{\cancel{v_0}\cancel{\sqrt{2}}}{\cancel{2}}\left(\frac{200\cancel{\sqrt{2}}}{\cancel{v_0}}\right) - \frac{g\left(\frac{200\sqrt{2}}{v_0}\right)^2}{2} = 100$

$200 - \frac{\frac{g\,40000\cdot\cancel{2}}{v_0^2}}{\cancel{2}} = 100$

$\frac{100}{1} = \frac{40000g}{v_0^2}$

$\sqrt{v_0^2} = \sqrt{\frac{40000g}{100}}$

$v_0 = \frac{200\sqrt{g}}{10}$

$v_0 = 20\sqrt{g}$

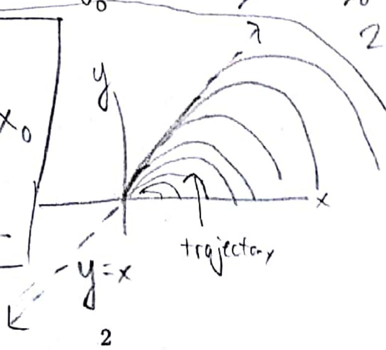
$v_0 = 20 \cdot 3.1 = 62$

$\boxed{v_0 = 62}$ 1

$200 \times 200 = 40000$

$3.1 \times 20 = 62.0$

iii) if v_0 can vary, then all points where $y_0 < x_0$ can be reached and $x_0 > 0$



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Problem 3. (4)

(i)Find $\lim_{(x,y)\mapsto(1,2)} \frac{(x-1)(y-2)^2+(x-1)^2(y-2)}{(x-1)^2+(y-2)^2}$.

(ii) Function f is defined as

$$f(x,y)=\begin{cases} \frac{(x-1)(y-2)^2+(x-1)^2(y-2)}{(x-1)^2+(y-2)^2}, & when\,(x,y)\neq(1,2) \\ 1, & when\,(x,y)=(1,2) \end{cases}$$

Find all points in $\mathbb{R}^2$ where f is continuous.

i) $\lim_{(x,y)\to(1,2)} \frac{(x-1)(y-2)^2+(x-1)^2(y-2)}{(x-1)^2+(y-2)^2}$

substitute $u=x-1$
$v=y-2$

$\lim_{(u,v)\to(0,0)} \frac{uv^2+u^2v}{u^2+v^2}$

$= \lim_{(u,v)\to(0,0)} \frac{uv^2}{u^2+v^2} + \lim_{(u,v)\to(0,0)} \frac{u^2v}{u^2+v^2}$ substitute $u=r\cos\theta$, $v=r\sin\theta$

$= \lim_{r\to0} \frac{r^{\cancel{3}}\cos\theta\sin^2\theta}{\cancel{r^2}(\cos^2\theta+\sin^2\theta)} + \lim_{r\to0} \frac{r^{\cancel{3}}\cos^2\theta\sin\theta}{\cancel{r^2}(\cos^2\theta+\sin^2\theta)}$

$= \lim_{r\to0} r\cos\theta\sin^2\theta + \lim_{r\to0} r\cos^2\theta\sin\theta$

$(-)$ -.5 $0+0$

$= \boxed{0}$

$\lim_{(x,y)\to(1,2)}$ . . .

ii) f is continuous at all points in $\mathbb{R}^2 \neq (1,2)$
since at $(1,2)$, $\lim_{(x,y)\to(1,2)} f(x,y) = 0 \neq f(1,2)=1$

and at all other points f is a rational function ✓
which is always continuous as long as the denominator $\neq 0$.

by the limit rules: $f_1(x)=x-1$ is continuous
and $f_2(y)=y-2$ is continuous

$\therefore f(x,y) = \frac{(x-1)(y-2)^2+(x-1)^2(y-2)}{(x-1)^2+(y-2)^2}$ is continuous where $(x-1)^2+(y-2)^2 \neq 0$
thus $(x,y)\neq(1,2)$

3

Problem 4. (4)

Let $f(x,y) = e^{x+y}\cos y$. $= e^x e^y \cos y$

(i) Find the equation of the tangent plane for $z = f(x,y)$ at point $x = 0, y = 0$.

(ii) Estimate the value $f(0.01, 0.01)$.

$f(0,0) = e^0 \cos 0 = 1$

$\partial_x f = e^y \cos y \, e^x$ at $(0,0) = 1$

$\partial_y f = (e^x e^y)(-\sin y) + (\cos y)(e^x e^y)$

at $(0,0) = \cancel{1 \cdot 1 \cdot (-\sin 0)} + \cos 0 \cdot 1 \cdot 1 = 1$

$z = L(x,y) = f(a,b) + \partial_x f(a,b)(x-a) + \partial_y f(a,b)(y-b)$

$= 1 + 1(x-0) + 1(y-0)$

$\boxed{z = 1 + x + y}$ ✓

ii) $f(0.01, 0.01) \approx L(0.01, 0.01)$

$= 1 + 0.01 + 0.01$

$= \boxed{1.02}$ ✓

Problem 5. (4)

Let $f(x, y, z) = x^2 + y^3 + z^4 - 10$.

(i) Find ∇f and $\nabla f(1, 2, 1)$. Find the direction at point $(1, 2, 1)$ in which the function f decreases most rapidly?

(ii) Find the directional derivative $D_{\mathbf{u}} f(1, 2, 1)$ where $\mathbf{u}$ is the direction of the vector $< 2, 1, 2 >$.

(iii) Find the equation for the tangent plane of the surface S defined by $x^2 + y^3 + z^4 = 10$ at the point $(1, 2, 1)$.

$\partial_x f = 2x \quad \partial_z f = 4z^3$

$\partial_y f = 3y^2$

i) $\nabla f = \langle \partial_x f, \partial_y f, \partial_z f \rangle$

$\boxed{\nabla f = \langle 2x, 3y^2, 4z^3 \rangle}$

1

$\nabla f(1,2,1) = \langle 2(1), 3(2)^2, 4(1)^3 \rangle = \boxed{\langle 2, 12, 4 \rangle}$

most rapid decrease $= -\nabla f(1,2,1) = \boxed{\langle -2, -12, -4 \rangle}$

ii) $D_{\vec{u}} f(1,2,1) = \dfrac{\vec{u}}{\|\vec{u}\|} \cdot \nabla f$

$\|\vec{u}\| = \sqrt{2^2 + 1^2 + 2^2} = \sqrt{9} = 3$

$= \frac{1}{3} \langle 2, 1, 2 \rangle \cdot \langle 2, 12, 4 \rangle$

$= \frac{1}{3}(4 + 12 + 8)$

$= \frac{1}{3}(24) = \boxed{8}$

1

iii) tan plane $= 0 = \partial_x f(a,b,c)(x-a) + \partial_y f(a,b,c)(y-b) + \partial_z f(a,b,c)(z-c)$

$0 = 2(x-1) + 12(y-2) + 4(z-1)$

$0 = 2x - 2 + 12y - 24 + 4z - 4$

-28
-2
-30

$\boxed{30 = 2x + 12y + 4z}$

1.5