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MATH 32A Midterm II, Fall 2017

Name: [redacted]

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Instruction: Justify all your answers. No Points Will Be Given Without Sufficient Reasoning/Calculations.

$\kappa = \frac{r' \times r''}{|r'|^3}$

$a_N = \kappa (v(t))^2$

$\vec{N} = \frac{a_N N}{\|a_N\|}$

$a_T = \vec{a} \cdot \vec{T}$

$v = \|v\|$

Problem 1. (4)

(i) Let $\mathbf{r}(t) = (bt, c\sin t, c\cos t)$ be the position vector of a moving particle, where b and c are positive constants. Find the velocity $\mathbf{v}$ and acceleration $\mathbf{a}$.

(ii) Find tangential and normal components of $\mathbf{a}$ in (i).

(iii) Let C be a curve given by $\mathbf{r}(t) = (bt, c\sin t, c\cos t)$ in (i) above. Using the results you obtained in (i) and (ii) above, find the curvature of C.

$\sqrt{t^2+c^2(1)}$

(i) $r(t) = \langle bt, c\sin t, c\cos t\rangle$

$\vec{v} = r'(t) = \langle b, c\cos t, -c\sin t\rangle$

$\vec{a} = r''(t) = \langle 0, -c\sin t, -c\cos t\rangle$

(ii) $a_T = \vec{a} \cdot \vec{T}$ $\quad \vec{T} = \frac{r'(t)}{|r'(t)|} = \frac{\langle t, c\cos t, -c\sin t\rangle}{\sqrt{t^2+c^2\cos^2 t + c^2\sin^2 t}}$

$a_N = \vec{a} \cdot N$

$a_T = \langle 1, -c\sin t, -c\cos t\rangle \cdot \left\langle \frac{t}{\sqrt{t^2+c^2}}, \frac{c\cos t}{\sqrt{t^2+c^2}}, \frac{-c\sin t}{\sqrt{t^2+c^2}}\right\rangle$ $\quad \vec{T} = \left\langle \frac{t}{\sqrt{t^2+c^2}}, \frac{c\cos t}{\sqrt{t^2+c^2}}, \frac{-c\sin t}{\sqrt{t^2+c^2}}\right\rangle$

$a_T = \frac{t}{\sqrt{t^2+c^2}} - \frac{c^2\sin t\cos t}{\sqrt{t^2+c^2}} + \frac{c^2\sin t\cos t}{\sqrt{t^2+c^2}} = \frac{t}{\sqrt{t^2+c^2}}$ $\quad \vec{N} = \frac{\vec{T}'(t)}{|\vec{T}'(t)|} =$

$a_N = \sqrt{|a|^2 - |a_T|^2} = \sqrt{(1+c^2\sin^2 t + c^2\cos^2 t) - \frac{t^2}{t^2+c^2}}$

$a_N = \sqrt{1+c^2(1) - \frac{t^2}{t^2+c^2}}$

$a_N = \sqrt{1+c^2 - \frac{t^2}{t^2+c^2}}$

correct given (i)

(iii) $a_N = \kappa(v(t)^2)$

$(\|v\|)^2 = \left(\sqrt{t^2+c^2\cos^2 t + c^2\sin^2 t}\right)^2$

$(\|v\|)^2 = \left(\sqrt{t^2+c^2}\right)^2$

$\|v\|^2 = t^2+c^2$

$\kappa(t^2+c^2) = \sqrt{1+c^2-\frac{t^2}{t^2+c^2}}$

$\kappa = \frac{\sqrt{1+c^2-\frac{t^2}{t^2+c^2}}}{t^2+c^2}$

$\cos(\pi/4)$

V_0

$\pi/4$

Problem 2. (4)

A bullet is fired from ground at an angle $\alpha = \pi/4$ with initial speed v_0.

(i)Find the position vector $\mathbf{r}(t)$, assuming that the initial position $\mathbf{r}(0) = (0,0)$.

(ii)Find the initial speed v_0 if the bullet hits the point $p = (200, 100)$.

(iii)Find all points $p = (x_0; y_0)$ that can be hit by the bullet, assuming that you can adjust the initial speed v_0 as you like.

Note : the gravity constant $g = 9.8m/s^2$ and the square root of g is (about) 3.1.

(i) $a = \langle 0, -g \rangle$ $r(0) = 0$

$\vec{v} = \int \vec{a}(t)\,dt = \langle 0, -gt \rangle + \langle V_0 \cos\alpha, V_0 \sin\alpha \rangle$

$\vec{r} = \int \vec{v}(t)\,dt = \langle 0, \frac{-gt^2}{2} \rangle + \langle V_0 \cos\alpha t, V_0 \sin\alpha t \rangle + \langle 0,0 \rangle$

$\vec{r} = \langle V_0 \cos(\frac{\pi}{4}) t, V_0 \sin(\frac{\pi}{4}) t - \frac{gt^2}{2} \rangle = \langle \frac{V_0\sqrt{2}t}{2}, \frac{V_0\sqrt{2}t}{2} - 4.9t^2 \rangle$

(i)

(ii) $V_0 \cos(\alpha) t = 200 \rightarrow t = \frac{200}{V_0 \cos(\alpha)}$

$V_0 \sin(\alpha) t - \frac{gt^2}{2} = 100$

$200 \tan\alpha$

$\tan(\pi/4) = 1$

$\frac{200 V_0 \sin\alpha}{V_0 \cos\alpha} - \frac{1}{2}(g)\left(\frac{200}{V_0 \cos\alpha}\right)^2 = 100$

$\cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$

$200(\tan\frac{\pi}{4}) - \frac{1}{2} g \left(\frac{40000}{V_0^2 \cos^2\alpha}\right) = 100$

$(\frac{\sqrt{2}}{2})^2 = \frac{2}{4} = \frac{1}{2}$

$\tan\frac{\pi}{4} = 1$

$-\frac{1}{2} g \left(\frac{40000}{V_0^2 \cos^2\alpha}\right) = -100$

$\cos^2(\pi/4) = (\frac{\sqrt{2}}{2})^2 = \frac{1}{2}$

$g\left(\frac{40000}{V_0^2 \cos^2\alpha}\right) = 200$

$40000g = 200(V_0^2 \cos^2\alpha)$

$40000g = 100(V_0^2)$

$V_0^2 = 400g$

$V_0 = \sqrt{400g} = 20(3.1)$

(1) $V_0 = 62 m/s$

(iii) $p = (x_0, y_0)$

Problem 3. (4)

(i)Find $\lim_{(x,y)\mapsto(1,2)} \frac{(x-1)(y-2)^2+(x-1)^2(y-2)}{(x-1)^2+(y-2)^2}$.

(ii) Function f is defined as

$$f(x,y) = \begin{cases} \frac{(x-1)(y-2)^2+(x-1)^2(y-2)}{(x-1)^2+(y-2)^2}, & when\ (x,y) \neq (1,2) \\ 1, & when\ (x,y) = (1,2) \end{cases}$$

Find all points in $\mathbf{R}^2$ where f is continuous.

(i) Polar
$x = r\cos\theta + 1$
$y = r\sin\theta + 2$

$\lim_{r\to 0} \frac{(r\cos\theta)(r\sin\theta)^2 + (r\cos\theta)^2(r\sin\theta)}{(r\cos\theta)^2 + (r\sin\theta)^2} = \frac{r^3\sin^2\theta\cos\theta + r^3(\sin\theta\cos^2\theta)}{r^2(\sin^2\theta + \cos^2\theta)}$

$\lim_{r\to 0}\left(\frac{r^3(\sin^2\theta\cos\theta + \sin\theta\cos^2\theta)}{r^2\ (1)}\right)$

$\lim_{r\to 0}\left(r(\sin^2\theta\cos\theta + \sin\theta\cos^2\theta)\right) = 0$

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(ii) $f(1,2) = 1$

f is continuous for all R^2 except at $(x,y) = (1,2)$

Because for a function to be continuous at a certain point, then $f(a,b) = \lim_{(x,y)\to(a,b)} f(x,y)$. However, we are given that $f(1,2) = 1$ but in part (i) we found that

$\lim_{(x,y)} \frac{(x-1)(y-2)^2 + (x-1)^2(y-2)}{(x-1)^2 + (y-2)^2} = 0$, and $1 \neq 0$,

So f is not continuous at $(1,2)$

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Problem 4. (4)

Let $f(x, y) = e^{x+y} \cos y$.

(i) Find the equation of the tangent plane for $z = f(x, y)$ at point $x = 0, y = 0$.

(ii) Estimate the value $f(0.01, 0.01)$.

$f_x = e^{x+y} \cos y$, $f_y = e^{x+y} \cos y - e^{x+y} \sin y$

$f(0,0) = e^{0+0} \cos(0) = 1$

$f_x(0,0) = e^{0+0} \cos(0) = 1$ $\quad f_y(0,0) = e^{0+0} \cos 0 - e^{0+0} \sin(0)$

$f_y(0,0) = 1$

(i)

$z = 1 + 1(x-0) + 1(y-0)$

$z = 1 + x + y$

$f(0.0, 0.01)$ plug into tangent plane

(ii)

$z = 1 + 0.01 + 0.01 = 1.02$

$f(0.01, 0.01) \approx 1.02$

Problem 5. (4)

Let $f(x, y, z) = x^2 + y^3 + z^4 - 10$.

(i) Find ∇f and $\nabla f(1, 2, 1)$. Find the direction at point $(1, 2, 1)$ in which the function f decreases most rapidly? $-\|\nabla f_p\|$

(ii) Find the directional derivative $D_{\mathbf{u}} f(1, 2, 1)$ where $\mathbf{u}$ is the direction of the vector $< 2, 1, 2 >$.

(iii) Find the equation for the tangent plane of the surface S defined by $x^2 + y^3 + z^4 = 10$ at the point $(1, 2, 1)$.

(i) $\nabla f = \langle 2x, 3y^2, 4z^3 \rangle$

$\nabla f(1,2,1) = \langle 2, 12, 4 \rangle$ 1.5

f decreases most rapidly when $-\|\nabla f_p\|$ $\quad -\|\nabla f(1,2,1)\| = -\sqrt{(2)^2+(12)^2+(4)^2}$

$-\|\nabla f(1,2,1)\| = -\sqrt{164}$

(ii) $\hat{u} = \dfrac{\langle 2,1,2 \rangle}{\sqrt{(2)^2+(1)^2+(2)^2}} = \langle \frac{2}{3}, \frac{1}{3}, \frac{2}{3} \rangle$

$D_u f(1,2,1) = \nabla f(1,2,1) \cdot \hat{u} = \langle 2,12,4 \rangle \cdot \langle \frac{2}{3}, \frac{1}{3}, \frac{2}{3} \rangle$

$D_u f(1,2,1) = \frac{4}{3} + \frac{12}{3} + \frac{8}{3} = \frac{24}{3} = 8$

1

(iii) $F(x,y,z) = x^2 + y^3 + z^4 - 10 = 0$

$\nabla F = \langle 2x, 3y^2, 4z^3 \rangle$

$\nabla F(1,2,1) = \langle 2, 12, 4 \rangle$

Tangent plane: $\quad 2(x-1) + 12(y-2) + 4(z-1) = 0$

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