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MATH 32B Midterm I, Winter 2018

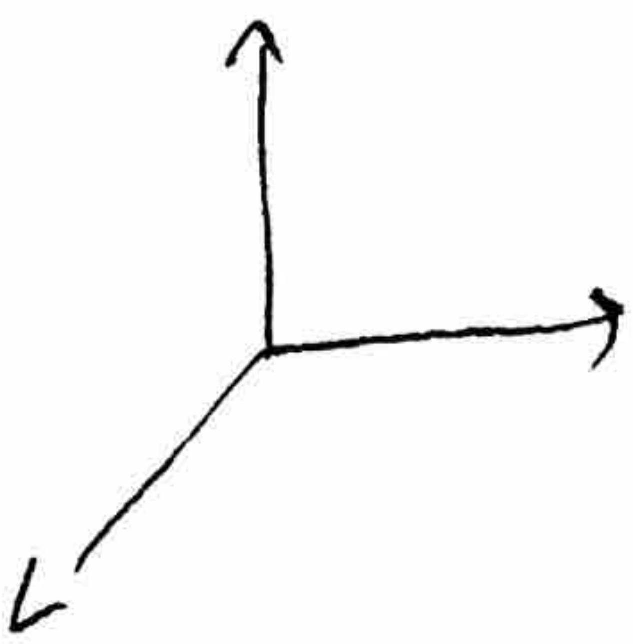
Name:

Circle Your TA's Name and Section Number: Eric Auld 2A 2B, Jacob Rooney 2E 2F, Alan Zhou 2C 2D

Justify All Your Answers. No Points Will Be Given Without Sufficient Reasoning/Calculations.

Problem 1. (4)

Find the triple integral $\int\int\int_E (x^2+z^2)\,dx\,dy\,dz$. Here E is the finite solid bounded by the surface $y = 4 - x^2 - z^2$ and the plane $y = 0$.



$0 = 4 - x^2 - z^2$

$x^2 + z^2 = 4$

$x = r\cos\theta \quad z = r\sin\theta$

$x^2 + z^2 = r^2$

$$\int_0^{2\pi}\int_0^2\int_0^{4-r^2} r^2 \cdot r \cdot dy\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 \left[r^3 y\right]_0^{4-r^2} dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 4r^3 - r^5\,dr\,d\theta = \int_0^{2\pi}\left[r^4 - \tfrac{1}{6}r^6\right]_0^2 d\theta$$

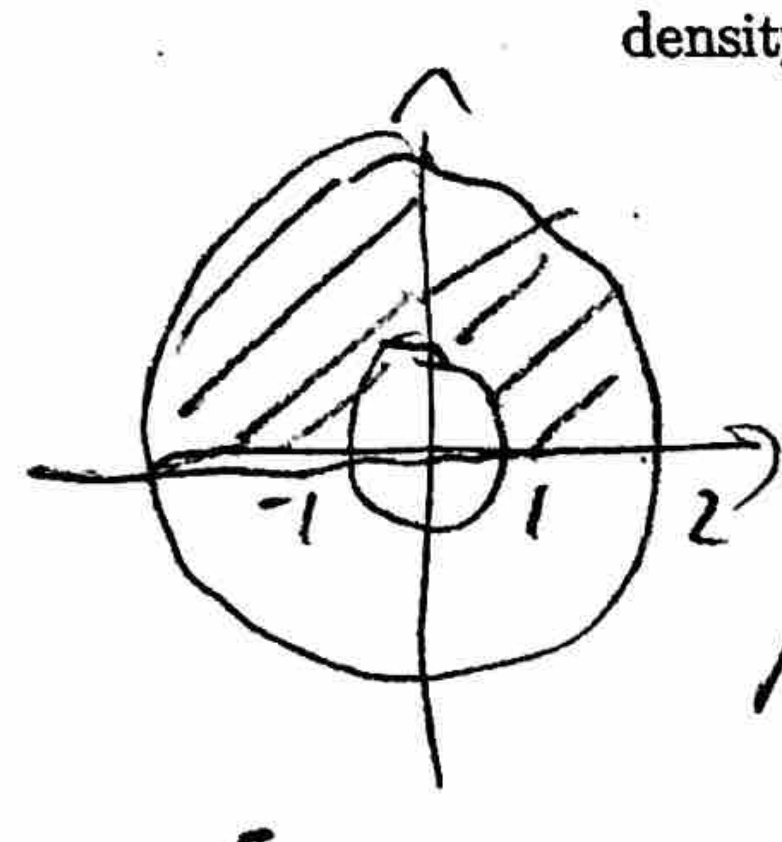
$$= \int_0^{2\pi} 16 - \frac{32}{3}\,d\theta = \int_0^{2\pi}\frac{16}{3}\,d\theta = \frac{16}{3}(2\pi) = \boxed{\frac{32\pi}{3}}$$

16 · 3 = 48

48 − 32 = 16

Problem 2. (4)

Compute the center of mass of $D = \{(x,y) \mid 1 \le x^2+y^2 \le 4,\ y \ge 0\}$. Here the density function $\rho(x,y) = y^2$.



$$M = \int_0^{\pi}\int_1^2 \rho(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

$$M = \int_0^{\pi}\int_1^2 r^2\sin^2\theta \cdot r\, dr\, d\theta$$

$$M = \int_0^{\pi}\int_1^2 r^3\sin^2\theta\, dr\, d\theta = \int_0^{\pi}\left[\frac{1}{4}r^4\sin^2\theta\right]_1^2 d\theta$$

$$= \int_0^{\pi} 4\sin^2\theta - \frac{1}{4}\sin^2\theta\, d\theta = \frac{15}{4}\int_0^{\pi}\sin^2\theta\, d\theta = \frac{15}{4}\int_0^{\pi}\frac{1}{2} - \frac{\cos 2\theta}{2}\, d\theta$$

$$= \frac{15}{4}\left[\frac{1}{2}\theta - \frac{\sin 2\theta}{4}\right]_0^{\pi} = \frac{15}{4}\left[\frac{\pi}{2} - 0 - (0-0)\right] = \frac{15\pi}{8} \quad 1.5$$

$$M_x = \int_0^{\pi}\int_1^2 r\sin\theta\, r^2\sin^2\theta\, r\, dr\, d\theta = \int_0^{\pi}\int_1^2 r^4\sin^3\theta\, dr\, d\theta$$

$$= \int_0^{\pi}\left[\frac{1}{5}r^5\sin^3\theta\right]_1^2 d\theta = \int_0^{\pi}\frac{32}{5}\sin^3\theta - \frac{1}{5}\sin^3\theta\, d\theta = \frac{31}{5}\int_0^{\pi}\sin^3\theta\, d\theta$$

$$= \frac{31}{5}\int_0^{\pi}(1-\cos^2\theta)\sin\theta\, d\theta = \frac{31}{5}\int_0^{\pi}\sin\theta - \sin\theta\cos^2\theta\, d\theta = \frac{31}{5}\left[-\cos\theta + \frac{1}{3}\cos^3\theta\right]_0^{\pi}$$

$$= \frac{31}{5}\left[1 - \frac{1}{3} - \left(-1 + \frac{1}{3}\right)\right] = \frac{31}{5}\left[\frac{2}{3} - \left(-\frac{2}{3}\right)\right] = \frac{31}{5}\cdot\frac{4}{3} = \frac{124}{15} \quad 1.5$$

$$M_y = \int_0^{\pi}\int_1^2 r\cos\theta\, r^2\sin^2\theta\, r\, dr\, d\theta = \int_0^{\pi}\int_1^2 r^4\sin^2\theta\cos\theta\, dr\, d\theta$$

$$= \int_0^{\pi}\left[\frac{1}{5}r^5\sin^2\theta\cos\theta\right]_1^2 d\theta = \int_0^{\pi}\frac{32}{5}\sin^2\theta\cos\theta - \frac{1}{5}\sin^2\theta\cos\theta\, d\theta$$

$$= \frac{31}{5}\int_0^{\pi}\sin^2\theta\cos\theta\, d\theta = \frac{31}{5}\left[\frac{1}{3}\sin^3\theta\right]_0^{\pi} = \frac{31}{5}(0) = 0$$

because moments about y are symmetric (region is symmetric over y axis and so is density function)

$$\bar{x} = \frac{M_y}{M} = 0 \qquad \bar{y} = \frac{M_x}{M} = \frac{\frac{124}{5}}{\frac{15\pi}{8}} = \frac{124\cdot 8}{5\cdot 15\pi} = \frac{992}{75\pi}$$

Center of Mass $\left(0, \frac{992}{75\pi}\right)$

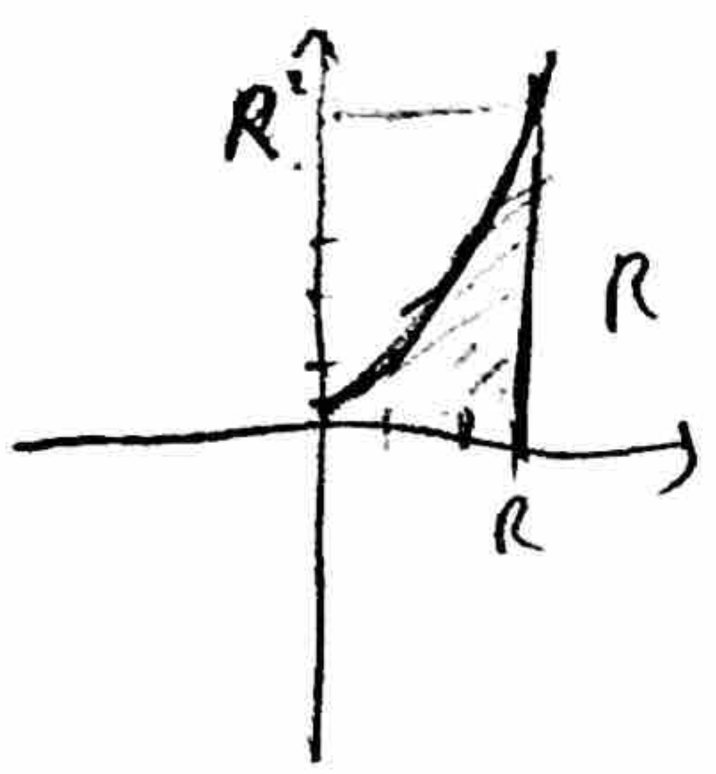
$$\begin{array}{r} 124 \\ \times\ 8 \\ \hline 992 \end{array}$$

$y = x^2$

Problem 3. (4)

Find the iterated integral $\int_0^{R^2} \int_{x=\sqrt{y}}^{R} y cos(x^5) dx dy$. Here R is a positive constant.

Hint: Convert the iterated integral into a double integral and evaluate the double integral.



$$\int_0^{R^2}\int_{x=\sqrt{y}}^{R} y\cos(x^5)\,dx\,dy$$

$$= \int_0^R \int_0^{x^2} y\cos(x^5)\,dy\,dx$$

$$= \int_0^R \left[\frac{1}{2}y^2\cos(x^5)\right]_0^{x^2} dx$$

$$= \int_0^R \frac{1}{2}x^4\cos(x^5)\,dx$$

$$= \frac{1}{2}\int_0^R \frac{1}{5}\cos(u)\,du \qquad u = x^5,\ du = 5x^4\,dx,\ \frac{1}{5}du = x^4\,dx$$

$$= \frac{1}{2}\left[\frac{1}{5}\sin(x^5)\right]_0^R$$

$$= \frac{1}{2}\left(\frac{1}{5}\sin(R^5) - 0\right)$$

$$= \boxed{\frac{1}{10}\sin(R^5)}$$

$1.5 + 1.5 = 3$

Problem 4. (4)

Find

$$\iiint_E x^2 + y^2 \, dx dy dz,$$

where E is the finite solid bounded by the sphere $x^2 + y^2 + z^2 = -2z$ and the cone $z = -\sqrt{x^2 + y^2}$

Note : E is inside both the sphere and the cone.

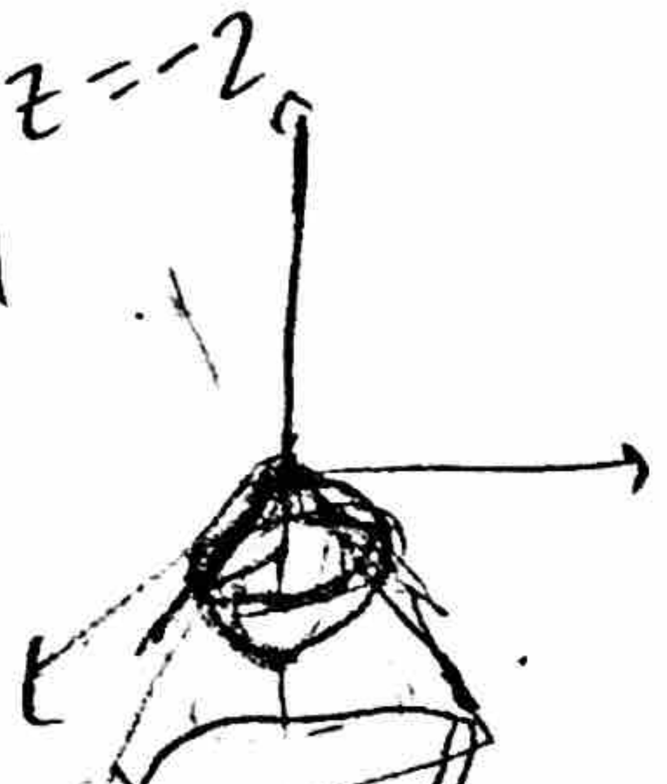
$\rho^2 = -2\rho\cos\phi$

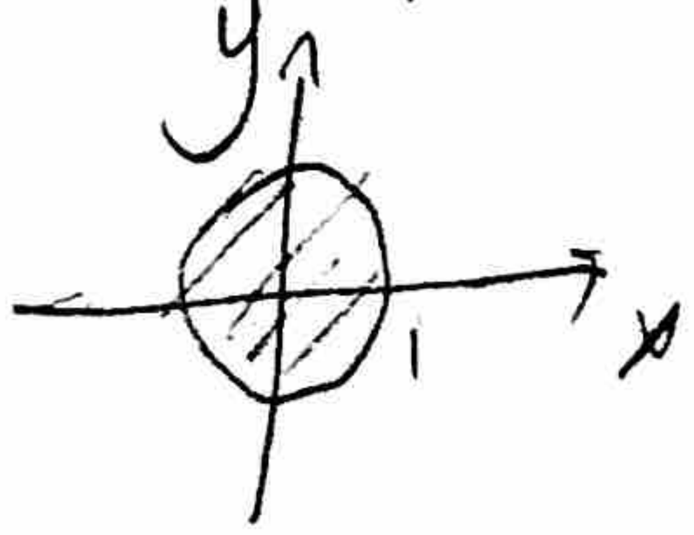
$\rho = -2\cos\phi$

$2z^2 = -2z$

$2z = -2$

$z = -1$





$z = -\sqrt{r^2}$

$z = -r$

$-1 - \sqrt{1-r^2} \le z \le -r$

$0 \le r \le 1$

$0 \le \theta \le 2\pi$

$x^2 + y^2 + (z+1)^2 = 1$

$(z+1)^2 = 1 - x^2 - y^2$

$(z+1)^2 = 1 - r^2$

$z + 1 = \pm\sqrt{1-r^2}$

$z = -1 - \sqrt{1-r^2}$

$$\int_0^{2\pi}\int_0^1\int_{-1-\sqrt{1-r^2}}^{-r} (r^2) r \, dz dr d\theta$$

nothing in this attempt is worth more than the below

$$= \int_0^{2\pi}\int_0^1 [r^3 z]_{-1-\sqrt{1-r^2}}^{-r} dr d\theta = \int_0^{2\pi}\int_0^1 -r^4 - (r^3(-1-\sqrt{1-r^2})) \, dr d\theta$$

$$= \int_0^{2\pi}\int_0^1 -r^4 - (-r^3 - r^3\sqrt{1-r^2}) dr d\theta = \int_0^{2\pi}\int_0^1 -r^4 + r^3 + r^3\sqrt{1-r^2} \, dr d\theta$$

$u = 1 - r^2$ $\quad r^2 = 1 - u$

$du = -2r dr$

$-\frac{1}{2} du = r dr$

$$= \int_0^{2\pi} \left[-\frac{1}{5}r^5 + \frac{1}{4}r^4 - \frac{1}{2}(1-u)\sqrt{u}\right] d\theta$$

$$\int_0^{2\pi}\int_{\frac{3\pi}{4}}^{\pi}\int_0^{-2\cos\phi} \rho^2 \sin^2\phi \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_{\frac{3\pi}{4}}^{\pi} \left[\frac{1}{5}\rho^5 \sin^3\phi\right]_0^{-2\cos\phi} d\phi d\theta = \int_0^{2\pi}\int_{\frac{3\pi}{4}}^{\pi} \frac{-32\cos^5\phi}{5} \sin^3\phi \, d\phi d\theta$$

$$= \int_0^{2\pi}\int_{\frac{3\pi}{4}}^{\pi} \frac{-32}{5}(\cos^5\phi - \cos^7\phi)\sin\phi \, d\phi = -\frac{32}{5}\int_0^{2\pi}\left[\frac{1}{6}\cos^6\theta - \frac{1}{8}\cos^8\theta\right]_{\frac{3\pi}{4}}^{\pi} d\theta$$

$$= -\frac{32}{5} \cdot 2\pi \left[\frac{1}{6}\cos^6\theta - \frac{1}{7}\cos^7\theta\right]_{\frac{3\pi}{4}} =$$

Problem 5. (4)

Estimate the following integral

$$\iiint_E e^{\cos(x^2+y^2)\cdot \sin z}\, dV,$$

where E is the solid inside the cylinder $x^2+y^2=R^2$ with $0 \le z \le R$. Here R is a positive constant.

convert to cylindrical

$$\int_0^{2\pi}\int_0^R\int_0^R r e^{\cos(r^2)\sin z}\, dz\, dr\, d\theta$$

Since $\cos(r^2)\sin z$ has max value 1 and min value -1, then $re^{-1} \le re^{\cos(r^2)\sin(z)} \le re$

$$\int_0^{2\pi}\int_0^R\int_0^R re^{-1}\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^R \left[zre^{-1}\right]_0^R dr\, d\theta = \int_0^{2\pi}\int_0^R Rre^{-1}\, dr\, d\theta$$

$$= \int_0^{2\pi}\left[\frac{1}{2}Rr^2e^{-1}\right]_0^R d\theta = \int_0^{2\pi}\frac{1}{2}R^3e^{-1}\, d\theta = \frac{\pi R^3}{e}$$

$$\int_0^{2\pi}\int_0^R\int_0^R re\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^R [zre]_0^R\, dr\, d\theta = \int_0^{2\pi}\int_0^R Rre\, dr\, d\theta$$

$$= \int_0^{2\pi}\left[\frac{1}{2}r^2Re\right]_0^R d\theta = \int_0^{2\pi}\frac{1}{2}R^3e\, d\theta = \pi R^3 e$$

so $$\frac{\pi R^3}{e} \le \iiint_E e^{\cos(x^2+y^2)\sin z}\, dV \le \pi R^3 e$$