1	4
2	1
3	2
4	4
5	0
T	11

MATH 32A Midterm I, Fall 2017

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Justify All Your Answers. No Points Will Be Given Without Sufficient Reasoning/Calculations.

Problem 1. (4)

Let $\mathbf{a} =< 1, 2, 3 >$, $\mathbf{b} =< 1, 2, 0 >$, $\mathbf{c} =< 1, -2, 4 >$.

(i) Find $\mathbf{b} \times \mathbf{c}$.

(ii) Find the area of the parallelogram spanned by $\mathbf{b}$ and $\mathbf{c}$ and the volume of the parallelepiped spanned by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

$|b\times c|$ $\quad |a\circ(b\times c)|$

(i) $$b\times c = \begin{vmatrix} i & j & k \\ 1 & 2 & 0 \\ 1 & -2 & 4 \end{vmatrix} = i\left(4(2)-(-2)(0)\right) - j\left(4(1)-1(0)\right) + k\left(-2(1)-1(2)\right)$$
$$= 8i - 4j - 4k$$
$$= \langle 8, -4, -4 \rangle$$

(ii) $$\text{area} = |b\times c| = \sqrt{(8)^2+(-4)^2+(-4)^2} = \sqrt{96} \text{ units}^2$$

volume of parallelepiped $= |a\cdot(b\times c)|$
$$= \langle 1,2,3\rangle\cdot\langle 8,-4,-4\rangle = |1(8)+(2)(-4)+3(-4)|$$
$$= |-12| = 12$$

Volume $= 12$ units^3

Problem 2. (4) L is parallel to planes $x+2y=1$ $x-2y+4z=1$

Find an equation for the line L which passes through the point $P_0 = (1, 2, 1)$ and is ~~perpendicular~~ parallel to both planes $x + 2y = 1$ and $x - 2y + 4z = 1$.

$v \cdot \underset{n_1}{\langle 1, 2, 0\rangle} = 0$ we want

Direction vector? we need one that's perpendicular to these 2 normal vectors

$v \cdot \underset{n_2}{\langle 1, -2, 4\rangle} = 0$ $v \perp n_1 \,\&\, n_2$

$v = n_1 \times n_2$

$v = \langle 8, -4, 4\rangle$

$L(t) = (1, 2, 1) + t\langle a, b, c\rangle$

+1

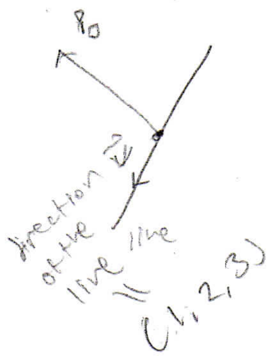
$L = (1,2,1) + t\,(8, -4, -4)$

part ① the line $(t, 2t+1, 3t-1) \subseteq M$

$\langle 1,2,3\rangle$ is parallel to M

$(1,3,2) \in M$ $\quad M \in (1,1,3)$

$(1,3,2)-(1,1,3) = (0,2,-1)$ parallel to M



$$\vec{n} = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 0 & 2 & -1 \end{vmatrix} = -8, 1, 2$$

$8x - y - 2z = d$

Problem 3. (4)

(i) Find the equation for plane M satisfying the following two conditions: (1) line $x = t$, $y = 2t+1$, $z = 3t-1$ is contained in M and (2) point $P_0 = (1,1,3)$ is contained in M.

(ii) Find the equation for the plane containing the line $x = t$, $y = 2t+1$, $z = 3t-1$ and perpendicular to the plane $x + 2y + z = 4$.

(i) let $P_1 = \vec{r}(0) = \langle 0, 1, -1\rangle$ $\qquad \vec{r}(t) = \langle t, 2t+1, 3t-1\rangle$

$\overrightarrow{P_0P_1} = (1,1,3) - (0,1,-1) = (1,0,4)$

let $P_2 = \vec{r}(1) = \langle 1, 3, 2\rangle$

$\overrightarrow{P_0P_2} = (1,1,3) - (1,3,2) = (0,-2,1)$

$$\vec{n} = \overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2} = \begin{vmatrix} i & j & k \\ 1 & 0 & 4 \\ 0 & -2 & 1 \end{vmatrix} = i(1(0) - (-2)(4)) - j(1(1) - 0(4)) + k(-2(1) - 0(0))$$
$$= \langle 8, -1, -2\rangle$$

Plane $M = 8(x-1) - 1(y-1) - 2(z-3) = 0$ $\quad$ 2

(ii) $\vec{r}(t) = (t, 2t+1, 3t-1)$ $\qquad \langle t, 2t+1, 3t-1\rangle \cdot \langle 1,2,1\rangle = 0$

~~$\vec{r}(0) = \langle 0,1,3\rangle$~~ $\qquad t + 4t + 2 + 3t - 1 = 0$

$8t = -1$

$t = -\frac{1}{8}$

Plane $= -\frac{1}{8}(x-0) + \frac{3}{4}(y-1) - \frac{11}{8}(z-3) = 0$ $\qquad$ ~~$\vec{r}(-\frac{1}{8}) = (-\frac{1}{8}, -\frac{1}{4}+1, -\frac{3}{8}-1)$~~

~~$\vec{r}(-\frac{1}{8}) = \langle -\frac{1}{8}, \frac{3}{4}, -\frac{11}{8}\rangle$~~

(ii) $L \in P$ $\qquad$ point on the plane & a normal vector

$L = (t, 2t+1, 3t-1)$

$\perp$ to the plane $(x+2y+z=4)$

Direction vector $V = \langle 1,2,3\rangle$

Normal vector for desired plane $V \times V_1 = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 1 & 2 & 1 \end{vmatrix} = \langle -4, 2, 0\rangle$

Plane $x + 2y + z = 4$

normal vector $= \langle 1,2,1\rangle$

V_1

$-4(x-0) + 2(y-1) + 0(z+1) = 0$

$2x - y = d$

$2x - y = -1$

$2x - y + 1 = 0$ $\qquad \vec{r}(0) = (0,1,-1)$

$(1-u^2)^{1/2}$
$\frac{1}{2}(1-u^2)^{-1/2}(2u)$

Problem 4. (4)

(i) Let $\mathbf{r}(u) = < u, e^u, \sqrt{1-u^2} >$. Find $\frac{d\mathbf{r}}{du}$.

(ii) Let $f(t) = \cos t$ and define $\mathbf{r}_1(t) = \mathbf{r}(f(t))$ with $\mathbf{r}$ in (i) above and $t \in [0, \pi]$. Find $\frac{d\mathbf{r}_1}{dt}$ in two ways: (A) do it directly (B) use the chain rule.

(iii) Let C be the curve given by the vector-valued function $\mathbf{r}_1(t)$ in (ii) above. Find the tangent line of C at the point $\mathbf{r}_1(\frac{\pi}{2})$.

$(1-u^2)^{1/2}$
$2u\frac{1}{2}(1-u^2)^{-1/2}$

$\sqrt{1-(\cos t)^2} = \sqrt{(\sin t)^2} = \sin t$

(i) $\frac{dr}{du} = \langle 1, e^u, \frac{-u}{\sqrt{1-u^2}} \rangle$

(ii) (A) $\vec{r}_1(t) = \langle \cos t, e^{\cos t}, \sqrt{1-(\cos t)^2} \rangle = \langle \cos t, e^{\cos t}, \sin t \rangle$

$\vec{r}_1'(t) = \langle -\sin t, -\sin t\, e^{\cos t}, \cos t \rangle$

plug in $(\cos t)$ into $\vec{r}$ and find derivative

(B) chain rule: $r_1'(t) = r'(\cos t) \cdot (-\sin t)$

$f'(t) = -\sin t \qquad \vec{r}'(f(t)) = \langle 1, e^{\cos t}, \frac{\cos t}{\sqrt{1-(\cos t)^2}} \rangle$

$\frac{dr_1}{dt} = f'(t)\vec{r}'(f(t)) = -\sin t \langle 1, e^{\cos t}, \frac{-\cos t}{\sqrt{1-(\cos t)^2}} \rangle$

$= \langle -\sin t, e^{\cos t}, \cos t \rangle$

(iii) $r_1(\frac{\pi}{2}) = \langle \cos(\frac{\pi}{2}), e^{\cos t}, \sin t \rangle$

$r_1(\frac{\pi}{2}) = \langle 0, 1, 1 \rangle$

$\sqrt{\sin^2(t)}$
$\sqrt{\sin^2(\frac{\pi}{2})} = 1$
$e^{\cos(\frac{\pi}{2})} = 1$
$\cos(\frac{\pi}{2}) = 0$

Tangent line of $C = L(t)$

$L(t) = \langle 0, 1, 1 \rangle + t\langle -1, -1, 0 \rangle$

$\vec{r}_1'(\frac{\pi}{2}) = \langle -1, -1e^0, 0 \rangle$

$(= r_1(\pi/2) + t r'(\frac{\pi}{2})$

of a parametrized curve

$K(t) = \frac{\| r'(t) \times r''(t) \|}{\| r'(t) \|^3}$

$\frac{|f''(x)|}{(1+(f'(x))^2)^{3/2}}$

0

Problem 5. (4)

(i) What is the curvature of the circle $C : (x-1)^2 + (y-2)^2 = r^2$? Here r is a positive constant (~~justify your answer~~).

(ii) What is the curvature of the curve C given by $\mathbf{r} = < t^3, t+1, t^2 >$?

(iii) Find all points on the curve C in (ii) above where the curvature is zero.

$\frac{|r' \times r''|}{|r'|^3}$

(i)

(ii) $r = \langle t^3, t+1, t^2 \rangle$

$r' = \langle 3t^2, t, 2t \rangle$

$r'' = \langle 6t, 1, 2 \rangle$

$r' \times r'' = \begin{vmatrix} i & j & k \\ 3t^2 & t & 2t \\ 6t & 1 & 2 \end{vmatrix}$

$= \langle 0, 6t^2, -3t^2 \rangle$

$|r' \times r''| = \sqrt{(0)^2 + (6t^2)^2 + (-3t^2)^2}$

$= \sqrt{36t^4 + 9t^4}$

$|r' \times r''| = \sqrt{45t^4}$

$|r'| = \sqrt{9t^4 + t^2 + 4t^2}$

$= \sqrt{9t^4 + 5t^2}$

$k(t) = \frac{\sqrt{45t^4}}{(\sqrt{9t^4 + 5t^2})^3}$

(iii) $k(t) = 0$

(i) $r(t) = (1 + r\cos t, 2 + r\sin t)$

parametrize the $C: (x-1)^2 + (y-2)^2 = r^2$

$r'(t) = (-r\sin t, r\cos t) = r(-\sin t, \cos t)$

$r''(t) = r(-\cos t, -\sin t)$

$\| r'(t) \| = r$ because $\sin^2\theta + \cos^2\theta = 1$

$r'(t) \times r''(t) = r^2 \begin{vmatrix} \hat{i} & \hat{j} & k \\ -\sin t & \cos t & 0 \\ -\cos t & -\sin t & 0 \end{vmatrix} = \langle 0, 0, r^2 \rangle = r^2$

$\|$ b/c we pull out the r^2

$\text{curvature} = \frac{r^2}{r^3} = \frac{1}{r}$

curvature of a circle of radius R is $1/R$