Math 32A - Lecture 3: Final exam

Released: Monday 14th March at 8am PDT
Due: Tuesday 15th March by 8am PDT via Gradescope

The exam includes 12 pages (including this cover page) and 11 problems.

There are a total of 100 points available.

Please read the following instructions **very carefully** before starting the test:

• Solutions must be uploaded to Gradescope **before 8am PST on Tuesday March 15th**. *Absolutely no late submissions will be accepted. I encourage you to submit your solutions several hours before the deadline to allow for possible technical complications.*

• **Please start each problem on a new page**.

• Credit will be given for fully explaining your solutions, writing clearly, legibly, and in full English sentences.

• If any questions arise during the examination, please email me: forlano@math.ucla.edu

• The exam is open book: You are permitted to use notes, texts, computers, and standard internet sites (such as Wikipedia) during the exam. You may use any theorems or results from lectures, homeworks or the midterms, without proof. However, you must verify any required assumptions in using these results.

• **The work submitted must be entirely your own:** You may not collaborate or work with anyone else to complete the exam. Students are not permitted to use online human resources such as Chegg, Math Stack Exchange, etc.

• Violations of these rules will be regarded as academic dishonesty and will be reported to the office of the Dean of Students.

• Note that it may be possible to successfully answer parts of a multi-part question without answering earlier parts.

• Good luck!

1. (1 point) Please complete a cover page with:

   - Your full name.
   - Your University ID number.
   - The following statement, along with your signature and the date:

     "I assert, on my honor, that I have not received assistance of any kind from any other person while working on this exam and that I have not used any non-permitted materials or technologies during the period of this evaluation."

2. (8 points) Find the volume of the parallelepiped with adjacent edges PQ, PR and PS, where

$$P(3,0,1), \quad Q(-1,2,5), \quad R(5,1,-1), \quad S(0,4,2).$$

3. (a) (3 points) Find a parametrisation of the line of intersection of the planes $x - z = 1$ and $y + 2z = 3$

   (b) (5 points) Find the equation of the plane which contains the line of intersection from part (a) and is orthogonal to the plane $x + y - 2z = 1$.

4. Consider the space-curve $\mathbf{r}(t) = \langle \sin(2t), \cos(2t), t \rangle$ for $t \in \mathbb{R}$.

   (a) (3 points) Find the values of t for which the space-curve $\mathbf{r}$ is inside and on the sphere $x^2 + y^2 + z^2 = 5$.

   (b) (3 points) Using your result from part (a), find the arc-length of the portion of the curve inside the sphere.

   Note: If you did not complete part (a), just find the arc-length from $t = 0$ to $t = 100$.

5. (10 points) Determine the equation of the osculating circle to $y = x^4 - 2x^2$ at $x = 1$.

6. (10 points) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \frac{\log(|x| + 1 - y)\sqrt{2 - x^2 - (y - 1)^2}}{\sqrt{y + x^2 - 1}}.$$

Determine and sketch the maximal domain of f.

7. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = e^{x\sqrt{y}}$.

   (a) (5 points) Use a linear approximation to $f(x, y)$ to estimate the value of $f(0.95, 4.04)$.

   (b) (3 points) Find the directional derivative of f at the point $(1, 4)$ in the direction of $\langle -1, -2 \rangle$.

   (c) (2 points) Find the maximum rate of change of f at the point $(1, 4)$.

8. (12 points) Let $q = q(t, x)$ be a solution to the linear wave equation

$$\frac{\partial^2 q}{\partial t^2} = c^2 \frac{\partial^2 q}{\partial x^2},$$

where $(t, x) \in \mathbb{R}^2$ and $c \neq 0$ is a constant. Define $u = x + ct$, $v = x - ct$ and

$$r(u, v) = q(t(u, v), x(u, v)).$$

Show that r solves

$$\frac{\partial^2 r}{\partial u \partial v} = 0.$$

Note: The equation in terms of r is now easily solvable!

9. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} \dfrac{x^3 y}{x^6 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

(a) (6 points) Show that the directional derivative $D_{\mathbf{u}} f(0, 0)$ exists for all unit vectors $\mathbf{u}$.

(b) (4 points) Show that f is not continuous at $(0, 0)$.

(c) (3 points) Where is f differentiable on $\mathbb{R}^2$? Explain your reasoning.

10. (12 points) Find and classify the local extrema of the function

$$f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2.$$

11. (10 points) Use Lagrange multipliers to find the maxima and minima of the function $f(x, y) = x^2y$ on the curve $x^4 + y^2 = 1$.