Math 32A
Fall 2016
Midterm 1
November 7th, 2016

Name:

UID: [redacted]

Section:(circle one) 3A (Tue) 3B (Thu) w/Ioannis
3C (Tue) 3D (Thu) w/John
3E (Tue) 3F (Thu) w/Tianqi

This exam contains 5 pages (including this cover page) and 4 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you make use of a theorem from lecture (or the textbook) in the course of your work, make sure to indicate which theorem was used and how it was used. Failure to do so will result in deduction of points.
- Write your solutions in the space below the questions. If you need more space, use the back of the page and clearly indicate when you have done this. Do not turn in your scratch paper.

Do not write in the table to the right.

Problem	Points	Score
1	20	20
2	20	20
3	20	13
4	20	20
Total:	80	73

1. (20 points) Consider the curve C parameterized by

$$\mathbf{r}(t) = \langle \cos(4t), \sin(4t), 3t \rangle$$

$\vec{r}\,'(t) = \langle -4\sin(4t), 4\cos(4t), 3 \rangle$

for $t \geq 0$.

(a) (5 points) Calculate the unit tangent vector $\mathbf{T}$ and unit normal vector $\mathbf{N}$ to the curve C at $\mathbf{r}(t)$ for $t \geq 0$.

$\|\vec{r}\,'(t)\| = \sqrt{16\sin^2 4t + 16\cos^2 4t + 9}$ since $\sin^2\theta + \cos^2\theta = 1$

$= \sqrt{16+9} = \sqrt{25} = 5$

• $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$

$= \dfrac{\langle -4\sin(4t), 4\cos(4t), 3 \rangle}{5} = \boxed{\langle -\frac{4}{5}\sin(4t), \frac{4}{5}\cos(4t), \frac{3}{5} \rangle}$

• $\vec{T}\,'(t) = \langle -\frac{16}{5}\cos(4t), -\frac{16}{5}\sin(4t), 0 \rangle \rightarrow \|\vec{T}\,'(t)\| = \sqrt{\frac{256}{25}\cos^2(4t) + \frac{256}{25}\sin^2(4t)} = \sqrt{\frac{256}{25}} = \frac{16}{5}$

$\vec{N}(t) = \dfrac{\vec{T}\,'(t)}{\|\vec{T}\,'(t)\|}$

$= \dfrac{\langle -\frac{16}{5}\cos(4t), -\frac{16}{5}\sin(4t), 0 \rangle}{\frac{16}{5}}$

$\vec{N}(t) = \boxed{\langle -\cos(4t), -\sin(4t), 0 \rangle}$

(b) (5 points) Calculate the arc length $s(t)$ of the parameterization $\mathbf{r}(t)$ as a function of t for $t \geq 0$.

$s(t) = \int_a^b \|\vec{r}\,'(u)\|\,du$ from part (a), $\|\vec{r}\,'(u)\| = 5$

$= \int_0^t \|\vec{r}\,'(u)\|\,du$

$s = \int_0^t 5\,du = 5t \rightarrow s = 5t$

(c) (5 points) Find the arc length parameterization $\mathbf{r}_1(s)$ of the curve C.

$s = 5t$ from part (b)

$g(t) = s(t) = 5t$

$g^{-1}(s) = t = \frac{s}{5}$ (inverse), plug in $\vec{r}(t)$

$\vec{r}_1(s) = \vec{r}(g^{-1}(s)) = \boxed{\langle \cos(\frac{4s}{5}), \sin(\frac{4s}{5}), \frac{3s}{5} \rangle}$

(d) (5 points) Calculate the curvature $\kappa(s)$ of the curve C at a point $\mathbf{r}_1(s)$.

$\kappa(t) = \|\frac{d\vec{T}}{ds}\|$

$\vec{r}_1(s) = \langle \cos(\frac{4s}{5}), \sin(\frac{4s}{5}), \frac{3s}{5} \rangle$

$\vec{r}_1\,'(s) = \langle -\frac{4}{5}\sin(\frac{4s}{5}), \frac{4}{5}\cos(\frac{4s}{5}), \frac{3}{5} \rangle$

$\|\vec{r}_1\,'(s)\| = 1$

$\vec{T}(s) = \frac{\vec{r}_1\,'(s)}{\|\vec{r}_1\,'(s)\|} = \langle -\frac{4}{5}\sin(\frac{4s}{5}), \frac{4}{5}\cos(\frac{4s}{5}), \frac{3}{5} \rangle$

Take derivative of T with respect to s

$= \langle -\frac{16}{25}\cos(\frac{4s}{5}), -\frac{16}{25}\sin(\frac{4s}{5}), 0 \rangle$ take magnitude

$= \sqrt{\frac{256}{625}\cos^2(\frac{4s}{5}) + \frac{256}{625}\sin^2(\frac{4s}{5}) + 0} = \sqrt{\frac{256}{625}} = \frac{16}{25}$ for all s

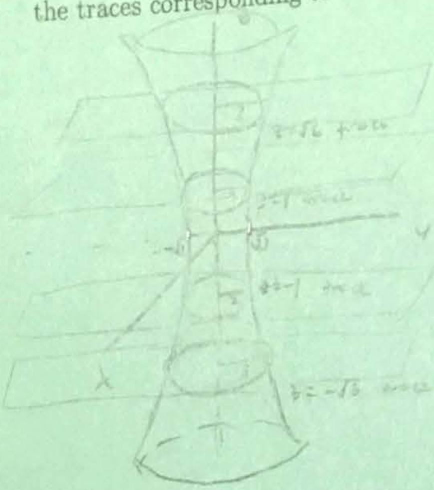
2. (20 points) Consider the quadratic surface given by the equation

$$x^2 + y^2 - z^2 = 3. \tag{1}$$

$x^2+y^2=4$

(a) (2 points) Classify the surface. That is, say what type of quadratic surface is given by Equation (1). $x^2+y^2=z^2+3$ $\boxed{\text{hyperboloid}}$, one sheet

(b) (8 points) Sketch the surface and, included in your sketch, draw the horizontal traces corresponding to $z = 1$, $z = -1$, $z = \sqrt{6}$ and $z = -\sqrt{6}$. Be sure to label your axes. You are free to draw additional traces (to help you in your sketch); however, you **must** label the traces corresponding to $z = \pm 1, \pm\sqrt{6}$.



- when $z=1$, $z=-1$, the horizontal trace is a circle, of radius 2
  $x^2+y^2=3+1 \to x^2+y^2=4$

- when $z=\pm\sqrt{6}$, the horizontal trace is a circle, of radius 3
  $x^2+y^2=3+6 \to x^2+y^2=9$

(c) (5 points) Give a parameterization $\mathbf{r}(t)$ of the trace corresponding to $z = \sqrt{6}$.

$x^2+y^2-z^2=3$

$x^2+y^2-(\sqrt{6})^2=3$

$x^2+y^2=9$

let $x=3\cos t$

$y=3\sin t$

$\boxed{\vec{r}(t)=\langle 3\cos t, 3\sin t, \sqrt{6}\rangle}$

(d) (5 points) Using your parameterization, calculate the curvature $\kappa(t)$ of the trace corresponding to $z = \sqrt{6}$ at $\mathbf{r}(t)$.

$\vec{r}(t)=\langle 3\cos t, 3\sin t, \sqrt{6}\rangle$ from part (c)

$\vec{r}\,'(t)=\langle -3\sin t, 3\cos t, 0\rangle$ $\quad \|\vec{r}\,'(t)\|=\sqrt{9}=3$

$\vec{r}\,''(t)=\langle -3\cos t, -3\sin t, 0\rangle$

$\vec{r}\,'(t)\times\vec{r}\,''(t)=\begin{vmatrix} i & j & k \\ -3\sin t & 3\cos t & 0 \\ -3\cos t & -3\sin t & 0\end{vmatrix} = 9\sin^2 t + 9\cos^2 t\, \hat{k}$

$= 9\hat{k} = \langle 0,0,9\rangle$

$\|\vec{r}\,'(t)\times\vec{r}\,''(t)\|=\sqrt{0+0+81}=9$

$\kappa(t)=\dfrac{\|\vec{r}\,'(t)\times\vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3}$

$=\dfrac{9}{3^3}$

$=\dfrac{9}{27}=\dfrac{1}{3}$

13/20

3. (20 points) Define a function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} \frac{y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases} \qquad \text{for } (x,y) \in \mathbb{R}^2.$$

(a) (8 points) Determine the set of points at which f is continuous. Justify your answer.

Let $\ell_1 = (0,t)$, $\ell_2 = (t,0)$

$(f \circ \ell_1)(t) = f(0,t) \rightarrow \lim_{t \to 0} f(0,t)$

$= \lim_{t \to 0} \frac{t}{t^2} = \frac{1}{t}$ limit does not exist at $(0,0)$ since $\lim_{t \to 0^+} \frac{1}{t} = \infty$ and $\lim_{t \to 0^-} \frac{1}{t} = -\infty$.

$(f \circ \ell_2)(t) = f(t,0) \rightarrow \lim_{t \to 0} f(t,0)$

$= \lim_{t \to 0} \frac{0}{t^2+0} = 0$

+6

Since $\lim_{(x,y) \to (0,0)} \frac{y}{x^2+y^2}$ DNE, then the function is not continuous at $(0,0)$.

However when, $D = \{(x,y) \in \mathbb{R}^2 : x \neq 0, y \neq 0\}$ the function is defined and is continuous at all points except $x=0$ and $y=0$

or

Thus, f is continuous in the domain $\boxed{D = \{(x,y) \in \mathbb{R}^2 : x \neq 0 \text{ and } y \neq 0\}}$

(b) (10 points) Compute the partial derivatives f_x and f_y where they exist.

$f(x,y) = \frac{y}{x^2+y^2}$, $x \neq 0$ and $y \neq 0$

$f_x = \frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) = \frac{0(x^2+y^2) - y(2x)}{(x^2+y^2)^2}$

$= \boxed{\frac{-2xy}{(x^2+y^2)^2}}$

+6

$f_y = \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) = \frac{1(x^2+y^2) - y(2y)}{(x^2+y^2)^2}$

$= \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}$

$= \boxed{\frac{x^2-y^2}{(x^2+y^2)^2}}$

(c) (2 points) Are the partial derivatives f_x and f_y continuous at all points in $\mathbb{R}^2$?

+1

f_x is not continuous at all points in $\mathbb{R}^2$, as function is not defined at $x=0, y=0$.

f_y is not continuous at all points in $\mathbb{R}^2$, as function is not defined at $x=0, y=0$

4. (20 points) In what follows, f is a two-variable function with domain $\mathcal{D} \subseteq \mathbb{R}$. You may assume that f is defined near $(0,0)$.

(a) (10 points) TRUE OR FALSE (circle one, 2 points each)

True	False	According to Kepler's laws, planets travel in ellipses with the sun at one focus.
True	False	The curvature κ of a curve C is *always* non-negative. ∂ or positive magnitude, ≥0.
True	False	To verify that the limit $\lim_{(x,y)\to(0,0)} f(x,y)$ exits and is equal to L, it *does not* suffice to show that $f(x,y)$ tends to L as (x,y) approaches $(0,0)$ along all lines of the form $y = mx$. what about x^2, may not exist
True	False	For f to be continuous at $(0,0)$, it is necessary that $(0,0) \in \mathcal{D}$. f is defined near that pt.
True	False	Contour lines corresponding to distinct z-values of f can intersect, but only at right angles.

(b) (5 points) Determine whether the following limit exists and, if it does, compute it. Justify your answer.

$$\lim_{(x,y)\to(0,0)} x^2 \sin\left(\frac{1}{x^4+y^2}\right)$$

squeeze theorem:

$-1 \le \sin\left(\frac{1}{x^4+y^2}\right) \le 1$ multiply by x^2 on all parts of the inequality to match above

$-x^2 \le x^2\sin\left(\frac{1}{x^4+y^2}\right) \le x^2$ take limit

$\lim_{(x,y)\to(0,0)} -x^2 \le \lim_{(x,y)\to(0,0)} x^2\sin\left(\frac{1}{x^4+y^2}\right) \le \lim_{(x,y)\to(0,0)} x^2$

$0 \le \lim_{(x,y)\to(0,0)} x^2\sin\left(\frac{1}{x^4+y^2}\right) \le 0$

By squeeze theorem, since both the left and right inequality is 0, it follows that

$\lim_{(x,y)\to(0,0)} x^2\sin\left(\frac{1}{x^4+y^2}\right) = \boxed{0}$

(c) (5 points) Consider the function

$$f(x,y) = \frac{\cos\left(\frac{x^4}{x^2+1}\right)}{2+\sin(x^2)} + x^2y^2$$

defined for all $(x,y) \in \mathbb{R}^2$. Calculate $f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$ at the point (a,b) (You may assume that f_{xy} and f_{yx} exist and are continuous on $\mathbb{R}^2$).

By assuming f_{xy} and f_{yx} exist and is continuous on $\mathbb{R}^2$, can use Clairaut's theorem where

$f_{xy} = f_{yx}$

f_{xy} is difficult, do $f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\cos\left(\frac{x^4}{x^2+1}\right)}{2+\sin(x^2)} + x^2y^2\right)\right)$

$= \frac{\partial}{\partial x}\left(2x^2y\right)$

$= \boxed{4xy}$