Math 32A
Fall 2016
Midterm 1
November 7th, 2016

Name: ____________________

UID: ____________________

Section:(circle one) 3A (Tue) 3B (Thu) w/Ioannis
3C (Tue) 3D (Thu) w/John
3E (Tue) 3F (Thu) w/Tianqi

This exam contains 5 pages (including this cover page) and 4 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you make use of a theorem from lecture (or the textbook) in the course of your work, make sure to indicate which theorem was used and how it was used. Failure to do so will result in deduction of points.
- Write your solutions in the space below the questions. If you need more space, use the back of the page and clearly indicate when you have done this. Do not turn in your scratch paper.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
Total:	80	

Do not write in the table to the right.

1. (20 points) Consider the curve $\mathcal{C}$ parameterized by

$$\mathbf{r}(t) = \langle 3t, \cos(4t), \sin(4t) \rangle$$

for $t \geq 0$.

(a) (5 points) Calculate the unit tangent vector $\mathbf{T}$ and unit normal vector $\mathbf{N}$ to the curve $\mathcal{C}$ at $\mathbf{r}(t)$ for $t \geq 0$.

We have

$$\mathbf{r}'(t) = \langle 3, -4\sin(4t), 4\cos(4t) \rangle \text{ and so } \|\mathbf{r}'(t)\| = \sqrt{3^2 + 4^2(\cos^2(4t) + \sin^2(4t))} = 5.$$

for all t. Consequently

$$\mathbf{T}(t) = (1/5)\langle 3, -4\sin(4t), 4\cos(4t) \rangle = \langle (3/5), -(4/5)\sin(4t), (4/5)\cos(4t) \rangle$$

for all t. Therefore

$$\mathbf{T}'(t) = \langle 0, -(4/5)4\cos(4t), (4/5)(-4)\sin(4t) \rangle = -(16/5)\langle 0, \cos(4t), \sin(4t) \rangle$$

for all t. Thus $\|\mathbf{T}'(t)\| = 16/5$ for all t. Consequently

$$\mathbf{N}(t) = -\frac{(16/5)}{(16/5)}\langle 0, \cos(4t), \sin(4t) \rangle = -\langle 0, \cos(4t), \sin(4t) \rangle$$

for all t.

(b) (5 points) Calculate the arc length $s(t)$ of the parameterization $\mathbf{r}(t)$ as a function of t for $t \geq 0$.

In view of our computations above,

$$s = s(t) = \int_0^t \|\mathbf{r}'(u)\|\, du = \int_0^t 5\, du = 5t$$

for all $t \geq 0$.

(c) (5 points) Find the arc length parameterization $\mathbf{r}_1(s)$ of the curve $\mathcal{C}$.

By the above computation, $s = g(t) = 5t$ and therefore $g^{-1}(s) = s/5$. Thus the arc length parameterization is given by

$$\mathbf{r}_1(s) = (\mathbf{r} \circ g^{-1})(s) = \mathbf{r}(g^{-1}(s)) = \mathbf{r}(s/5) = \langle 3s/5, \cos(4s/5), \sin(4s/5) \rangle$$

for $s \geq 0$.

(d) (5 points) Calculate the curvature $\kappa(s)$ of the curve $\mathcal{C}$ at a point $\mathbf{r}_1(s)$.

Because $\mathbf{r}_1(s)$ is an arc length parameterization of $\mathcal{C}$, the unit tangent vector for this parameterization (which is a different function than $\mathbf{T}(t)$ above) is given by

$$\mathbf{T}(s) = \mathbf{r}_1'(s) = \langle 3/5, -(4/5)\sin(4s/5), (4/5)\cos(4s/5) \rangle$$

and so, for $s \geq 0$,

$$\kappa(s) = \left\| \frac{d\mathbf{T}}{ds} \right\| = \|\langle 0, -(4/5)^2\cos(4s/5), -(4/5)^2\sin(4s/5)^2 \rangle\| = 16/25.$$

2. (20 points) Consider the quadratic surface given by the equation

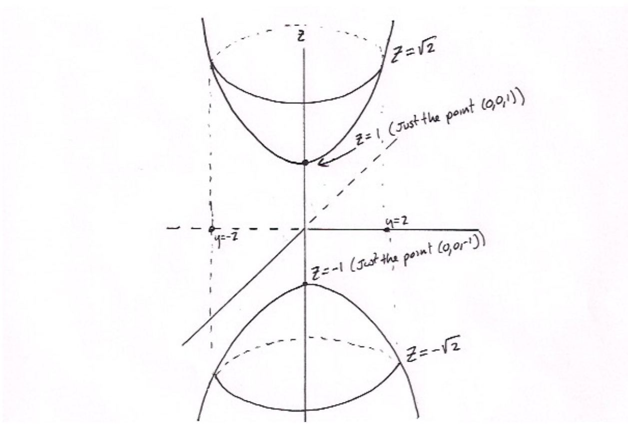
$$-x^2 - y^2 + 4z^2 = 4. \tag{1}$$

(a) (2 points) Classify the surface. That is, say what type of quadratic surface is given by Equation (1).
We can rewrite this as

$$-\left(\frac{x}{2}\right)^2 - \left(\frac{y}{2}\right)^2 + z^2 = 1$$

from which we see that the surface is a hyperboloid of two sheets.

(b) (8 points) Sketch the surface and, included in your sketch, draw the horizontal traces corresponding to $z = 1$, $z = -1$, $z = \sqrt{2}$ and $z = -\sqrt{2}$. Be sure to label your axes. You are free to draw additional traces (to help you in your sketch); however, you **must** label the traces corresponding to $z = \pm 1, \pm\sqrt{2}$.



(c) (5 points) Give a parameterization $\mathbf{r}(t)$ of the trace corresponding to $z = \sqrt{2}$ When $z = \sqrt{2}$, $-x^2 - y^2 = 4 - 4(\sqrt{2})^2 = -4$ and so $x^2 + y^2 = 4$. Thus, a parameterization of this trace is given by

$$\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t), \sqrt{2}\rangle$$

for $t \in \mathbb{R}$ or $t \in [0, 2\pi)$.

(d) (5 points) Using your parameterization, calculate the curvature $\kappa(t)$ of the trace corresponding to $z = \sqrt{2}$ at $\mathbf{r}(t)$.
We have, $\mathbf{r}'(t) = \langle -2\sin(t), 2\cos(t), 0\rangle$ and $\mathbf{r}''(t) = \langle -2\cos(t), -2\sin(t), 0\rangle$ for $t \in \mathbb{R}$. Thus

$$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\|\langle 0, 0, 4\rangle\|}{2^3} = \frac{1}{2}$$

for all $t \in \mathbb{R}$ because

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2\sin(t) & 2\cos(t) & 0 \\ -2\cos(t) & -2\sin(t) & 0 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + 4(\cos^2(t) + \sin^2(t))\mathbf{k} = \langle 0, 0, 4\rangle.$$

Another acceptable answer: The trace corresponding to $z = \sqrt{t}$ is a circle of radius $R = 2$ and therefore its curvature is a constant: $\kappa(t) = 1/R = 1/2$ for all t.

3. (20 points) Define a function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} \frac{x}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases} \qquad \text{for } (x,y) \in \mathbb{R}^2.$$

(a) (8 points) Determine the set of points at which f is continuous. Justify your answer.
Away from the origin f is a rational function whose denominator is non-vanishing. Hence f is continuous at all $(x,y) \in \mathbb{R}^2$ such that $(x,y) \neq (0,0)$. We must examine f at $(0,0)$. Observe that

$$\lim_{x\to 0^+} f(x,0) = \lim_{x\to 0} \frac{x}{x^2+0^2} = \lim_{x\to 0^+} \frac{1}{x} = \infty;$$

thus $f(x,y)$ grows without bound as (x,y) approaches $(0,0)$ along the positive x-axis. Equivalently

$$\lim_{t\to 0^+} f \circ \ell(t) = \lim_{t\to 0^+} f(t,0) = \infty$$

where $\ell(t) = (t,0)$ for $t \in \mathbb{R}$. Consequently, $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exists and therefore $f(x,y)$ is not continuous at $(0,0)$. Thus f is continuous on the set

$$\mathbb{R}^2 \setminus \{(0,0)\} = \{(x,y) \in \mathbb{R}^2 : (x,y) \neq (0,0)\}.$$

(b) (10 points) Compute the partial derivatives f_x and f_y where they exist.
For all $(x,y) \neq (0,0)$,

$$f_x(x,y) = \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

and

$$f_y(x,y) = \frac{0 - x(2y)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}.$$

Now, at $(x,y) = (0,0)$,

$$\lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{h}{h(h^2+0^2)} = \infty$$

and

$$\lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h(0^2+h^2)} = \lim_{h\to 0} 0 = 0.$$

Consequently, $f_y(0,0) = 0$ and $f_x(0,0)$ does not exist. Hence

$$f_x(x,y) = \frac{y^2-x^2}{(x^2+y^2)^2} \qquad \text{for } (x,y) \neq (0,0)$$

and

$$f_y(x,y) = \begin{cases} \frac{-2xy}{(x^2+y^2)^2} & \text{when } (x,y) \neq (0,0) \\ 0 & \text{when } (x,y) = (0,0). \end{cases}$$

(c) (2 points) Are the partial derivatives f_x and f_y continuous at all points in $\mathbb{R}^2$?
No. As f_x does not exist at $(0,0)$, it cannot be continuous at $(0,0)$. Though $f_y(0,0)$ does it exist, it isn't continuous at $(0,0)$; this can be seen by checking the limit along lines.

4. (20 points) In what follows, f is a two-variable function with domain $\mathcal{D} \subseteq \mathbb{R}^2$. You may assume that f is defined near $(0,0)$.

(a) (10 points) TRUE OR FALSE (circle one, 2 points each)

TRUE False		According to Kepler's laws, planets travel in ellipses with the sun at one focus.
TRUE False		The curvature κ of a curve $\mathcal{C}$ is *always* non-negative.
True **FALSE**		To verify that the limit $\lim_{(x,y)\to(0,0)} f(x,y)$ exits and is equal to L, it suffices to show that $f(x,y)$ tends to L as (x,y) approaches $(0,0)$ along all lines of the form $y = mx$.
TRUE False		For f to be continuous at $(0,0)$, it is necessary that $(0,0) \in \mathcal{D}$.
TRUE False		Contour lines corresponding to distinct z-values of f can *never* intersect.

(b) (5 points) Determine whether the following limit exists and, if it does, compute it. Justify your answer.

$$\lim_{(x,y)\to(0,0)} 4x^2y^4 \cos\left(\frac{1}{x^4+y^2}\right)$$

In view of the inequality $-1 \leq \cos(\theta) \leq 1$ for $\theta \in \mathbb{R}$, we have

$$-4x^2y^4 \leq 4x^2y^4 \cos\left(\frac{1}{x^4+y^2}\right) \leq 4x^2y^4$$

for all $(x,y) \neq (0,0)$. Because polynomials are continuous, we note that

$$\lim_{(x,y)\to(0,0)} -4x^2y^4 = 0 = \lim_{(x,y)\to(0,0)} 4x^2y^4.$$

Thus, an application of the Squeeze Theorem guarantees that

$$\lim_{(x,y)\to(0,0)} 4x^2y^4 \cos\left(\frac{1}{x^4+y^2}\right) = 0.$$

(c) (5 points) Consider the function

$$f(x,y) = x^2y + \frac{\cos\left(\frac{x^4}{x^2+1}\right)}{2+\sin(x^2)}$$

defined for all $(x,y) \in \mathbb{R}^2$. Calculate $f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$ at the point (a,b) (You may assume that f_{xy} and f_{yx} exist and are continuous on $\mathbb{R}^2$).
In view of the fact that the partial derivatives f_{xy} and f_{yx} exist and are continuous, we may apply Clairaut's theorem to conclude that

$$f_{xy}(a,b) = f_{yx}(a,b) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} f(x,y)\right)(a,b) = \frac{\partial}{\partial x}(x^2)(a,b) = 2a$$

where we have used the fact that $\frac{\partial}{\partial y}\left(\cos\left(\frac{x^4}{x^2+1}\right) / \left(2+\sin(x^2)\right)\right) = 0$