Math 32A
Fall 2016
Midterm 1
October 17th, 2016
Time Limit: 50 Minutes

Name:

UID:

Section:

This exam contains 5 pages (including this cover page) and 4 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

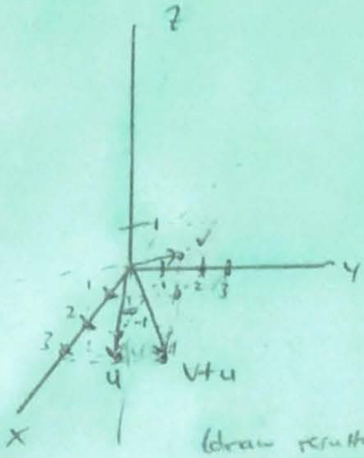
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- Write your solutions in the space below the questions. If you need more space, use the back of the page and clearly indicate when you have done this. Do not turn in your scratch paper.

Do not write in the table to the right.

Problem	Points	Score
1	25	25
2	25	24
3	25	23
4	25	25
Total:	100	97

1. Let $\mathbf{v} = \langle 1, 2, 1\rangle$ and $\mathbf{u} = \langle 2, 1, -1\rangle$.

(a) (5 points) Draw the vector $\mathbf{v} + \mathbf{u}$.



(draw resultant vector by parallelogram)

Double check: v+u should be $<3, 3, 0>$

(b) (10 points) Find the parallel projection of $\mathbf{u}$ onto $\mathbf{v}$, i.e., find the vector $\mathbf{u}_{||\mathbf{v}}$.

$$u_{||v} = \frac{u \cdot v}{||v||^2} v = \frac{<2,1,-1> \cdot <1,2,1>}{1^2+2^2+1^2} <1,2,1>$$

$$= \frac{2+2-1}{6} <1,2,1>$$

$$= \frac{3}{6} <1,2,1>$$

$$= \frac{1}{2} <1,2,1>$$

$$\boxed{u_{||v} = < \frac{1}{2}, 1, \frac{1}{2} >}$$

(c) (10 points) Find the area of the parallelogram spanned by $\mathbf{v}$ and $\mathbf{u}$.

$$A_{parallelogram} = ||v \times u||$$

$$= \sqrt{(-3)^2 + 3^2 + (-3)^2}$$

$$= \sqrt{9+9+9}$$

$$= \sqrt{27}$$

$$= \boxed{3\sqrt{3} \text{ units}^2}$$

$$v \times u = \begin{vmatrix} i & j & k \\ 1 & 2 & 1 \\ 2 & 1 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} i - \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} j + \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} k$$

$$= -3i + 3j + -3k$$

$$v \times u = <-3, 3, -3>$$

2. Given the points $P = (1, 2, 3)$, $Q = (3, 4, 4)$ and $R = (2, 2, 4)$, find:

$\|v\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$

(a) (5 points) The angle between $\overrightarrow{PQ}$ and $\overrightarrow{PR}$

$\overrightarrow{PQ} = \langle 3-1, 4-2, 4-3 \rangle = \langle 2, 2, 1 \rangle$

$\overrightarrow{PR} = \langle 2-1, 2-2, 4-3 \rangle = \langle 1, 0, 1 \rangle$

angle: $\overrightarrow{PQ} \cdot \overrightarrow{PR} = \|\overrightarrow{PQ}\| \|\overrightarrow{PR}\| \cos\theta$

$2(1) + 0 + 1(1) = \sqrt{4+4+1}\sqrt{1+1} \cos\theta$

$3 = \cancel{\sqrt{9}}^{3} \sqrt{2} \cos\theta$

$\cos\theta = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\cos\theta = \frac{\sqrt{2}}{2}$

$\boxed{\theta = \frac{\pi}{4}}$ $\cancel{\frac{7\pi}{4}}$ by convention angle is $0 \le \theta \le \pi$

(b) (10 points) A unit vector perpendicular to the plane containing P, Q and R

which is the normal vector

From part (c) $\hat{n} = \langle 2, -1, -2 \rangle$

unit vector $\vec{e}_n = \frac{\hat{n}}{\|\hat{n}\|} = \frac{\langle 2, -1, -2 \rangle}{\sqrt{2^2 + (-1)^2 + (-2)^2}} = \frac{\langle 2, -1, -2 \rangle}{\sqrt{9}} = \frac{1}{3}\langle 2, -1, -2 \rangle$

$\vec{e}_n = \boxed{\langle \frac{2}{3}, -\frac{1}{3}, -\frac{2}{3} \rangle}$

(c) (5 points) The equation of the plane containing P, Q and R

$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & 1 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}\hat{i} - \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}\hat{j} + \begin{vmatrix} 2 & 2 \\ 1 & 0 \end{vmatrix}\hat{k}$

$= 2\hat{i} - \hat{j} + -2\hat{k}$

$= \langle 2, -1, -2 \rangle$

choose point $P = (1, 2, 3)$

$\hat{n} \cdot \overrightarrow{P_0P} = 0$, $\overrightarrow{P_0P} = \langle x-1, y-2, z-3 \rangle$

$\langle 2, -1, -2 \rangle \cdot \langle x-1, y-2, z-3 \rangle = \boxed{2x - y - 2z = -6}$

(d) (5 points) The distance from the plane containing P, Q and R to the point $S = (0, 1, 0)$

4

choose arbitrary point in the plane, $T = (-3, 0, 0)$ that fits in the plane equation from (c).

$\overrightarrow{ST} = \langle 3, -1, 0 \rangle$ careful w/ signs.

Find distance of projection of $\overrightarrow{ST}$ onto $\hat{n}$ (normal vector) $\hat{n} = \langle 2, -1, -2 \rangle$

$\overrightarrow{ST}_{\parallel \hat{n}} = \frac{\overrightarrow{ST} \cdot \hat{n}}{\|\hat{n}\|^2}\hat{n} = \frac{\langle 3, -1, 0 \rangle \cdot \langle 2, -1, -2 \rangle}{2^2 + (-1)^2 + (-2)^2}\langle 2, -1, -2 \rangle$

$= \frac{6 + 1 + 0}{4 + 1 + 4}\langle 2, -1, -2 \rangle$

$\overrightarrow{ST}_{\parallel \hat{n}} = \frac{\cancel{7}\,5}{9}\langle 2, -1, -2 \rangle = \langle \frac{14}{9}, -\frac{7}{9}, -\frac{14}{9} \rangle$

$\|\overrightarrow{ST}_{\parallel \hat{n}}\| = \sqrt{\frac{196}{81} + \frac{49}{81} + \frac{196}{81}}$

$= \sqrt{\frac{441}{81}}$

$= \frac{\sqrt{147}}{\sqrt{27}}$

$= \frac{\sqrt{49}}{\sqrt{9}} = \boxed{\frac{\cancel{7}\,5}{3}}$ units

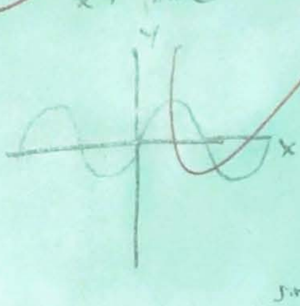
$3 \overline{)441}$ 147

$3 \overline{)147}$ 49

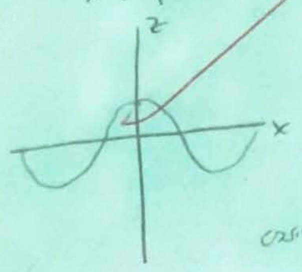
23/25

3. Consider the vector-valued function $\vec{\mathbf{r}}(t) = \langle t, \sin t, \cos t\rangle$ for $-\infty < t < \infty$.

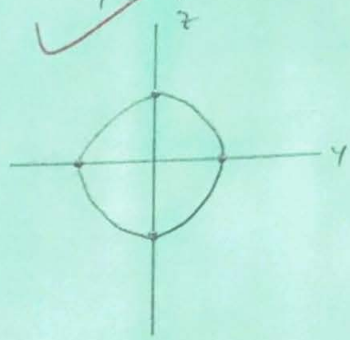
(a) (10 points) Draw the projections of $\vec{\mathbf{r}}(t)$ to the three coordinate planes. Using these, sketch the curve determined by $\vec{\mathbf{r}}(t)$.



t	$\sin t$
0	0
$\frac{\pi}{2}$	1
π	0
$\frac{3\pi}{2}$	-1
2π	0



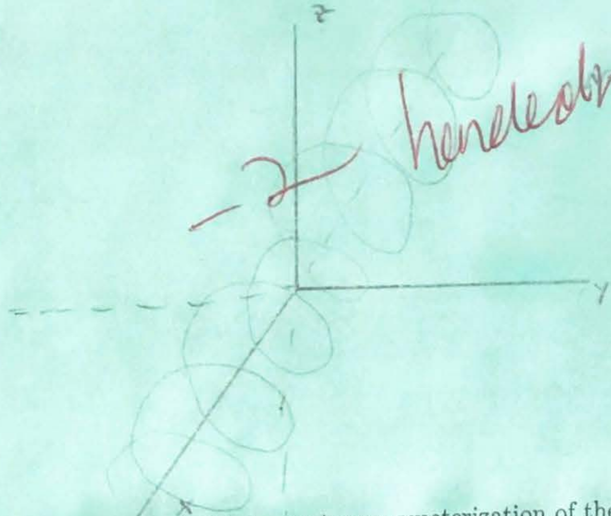
t	$\cos t$
0	1
$\frac{\pi}{2}$	0
π	-1
$\frac{3\pi}{2}$	0
2π	1



t	$\sin t$	$\cos t$
0	0	1
$\frac{\pi}{2}$	1	0
π	0	-1
$\frac{3\pi}{2}$	-1	0
2π	0	1

circle

8/10



-2 handedness

... of the circle, viewed at the yz plane is constant ... an past a sine drawn.

Putting all 3 projections together, the 3D graph makes a helix that travels throughout the x axis

(b) (15 points) Find a vector parameterization of the tangent line to $\vec{\mathbf{r}}(t)$ at $t = \pi/6$.

$\vec{r}(t) = \langle t, \sin t, \cos t\rangle$ $\quad \vec{r}(\frac{\pi}{6}) = \langle \frac{\pi}{6}, \frac{1}{2}, \frac{\sqrt{3}}{2}\rangle$

$\vec{r}\,'(t) = \langle 1, \cos t, -\sin t\rangle$ $\quad \vec{r}\,'(\frac{\pi}{6}) = \langle 1, \frac{\sqrt{3}}{2}, -\frac{1}{2}\rangle$

$\ell(t) = \vec{r}\,'(t)t + \vec{r}(t)$

tangent line at $t = \frac{\pi}{6}$ = $\boxed{\ell(\frac{\pi}{6}) = \langle 1, \frac{\sqrt{3}}{2}, -\frac{1}{2}\rangle t + \langle \frac{\pi}{6}, \frac{1}{2}, \frac{\sqrt{3}}{2}\rangle}$

$= \langle t + \frac{\pi}{6}, \frac{\sqrt{3}}{2}t + \frac{1}{2}, -\frac{1}{2}t + \frac{\sqrt{3}}{2}\rangle$

15/15

4. In answering the following question, recall that the zero vector is, by convention, orthogonal to every vector.

25

(a) (15 points) TRUE OR FALSE (circle one)

The dot product between two vectors is a ~~vector~~. *should be scaler*	TRUE	FALSE
The cross product between two vectors is a vector.	TRUE	FALSE
Two vectors $\mathbf{v}$ and $\mathbf{u}$ are orthogonal if and only if $\mathbf{v} \cdot \mathbf{u} = 0$.	TRUE	FALSE
For any three vectors $\mathbf{v}, \mathbf{u}$ and $\mathbf{w}$, $\mathbf{v} \times (\mathbf{u} \times \mathbf{w}) = (\mathbf{v} \times \mathbf{u}) \times \mathbf{w}$. *order matters, not associative!*	TRUE	FALSE
There exists a vector $\mathbf{v}$ such that $\mathbf{v} \times \langle 1,1,1 \rangle = \langle 1,2,0 \rangle$.	TRUE	FALSE

$\langle a,b,c \rangle \times \langle 1,1,1 \rangle = \langle 1,2,0 \rangle$

$\begin{vmatrix} i & j & k \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} =$

$(b-c)\hat{i} + (c-a)\hat{j} + (a-b)\hat{k}$

$b-c=1 \qquad c=b-1$

$c-a=2 \qquad b-1-a=2$

$a-b=0 \qquad b-a=3$

$-b+a=-3$

a impossible $a-b=-3$

(b) (10 points) Let $\vec{\mathbf{r}}(t)$ be differentiable and let C be a constant. Show that the following statement is true. Your reasoning/justification should be well-written and clear.

If $\|\vec{\mathbf{r}}(t)\| = C$ for all t, then $\frac{d}{dt}\vec{\mathbf{r}}(t) = \vec{\mathbf{r}}'(t)$ is orthogonal to $\vec{\mathbf{r}}(t)$ for all t.

$\|\vec{r}(t)\| = C$ for all t, now square it so we can do dot products with magnitudes

$\|\vec{r}(t)\|^2 = C^2$ Take derivative of both sides

$\frac{d}{dt}\|\vec{r}(t)\|^2 = \frac{d}{dt}C^2$, C is a constant so derivative of that constant is 0. (C^2 is a constant likewise)

$\frac{d}{dt}(\vec{r}(t)\cdot\vec{r}(t)) = 0 \qquad \|\vec{r}(t)\|^2 = \vec{r}(t)\cdot\vec{r}(t)$ by theorem

$\vec{r}\,'(t)\cdot\vec{r}(t) + \vec{r}(t)\cdot\vec{r}\,'(t) = 0$ expand, derivative of dot product.

$2\vec{r}(t)\cdot\vec{r}\,'(t) = 0$

$\vec{r}(t)\cdot\vec{r}\,'(t) = 0$

Since $\vec{r}(t)\cdot\vec{r}\,'(t) = 0$ that means that $\frac{d}{dt}\vec{r}(t) = \vec{r}\,'(t)$ is orthogonal to $\vec{r}(t)$, for all t. Hence, the statement is proved.

nice.