Math 32A
Fall 2016
Midterm 1
October 17th, 2016
Time Limit: 50 Minutes

Name: ______________________

UID: ______________________

Section: 3A 3B 3C 3D 3E 3F (circle one)

This exam contains 5 pages (including this cover page) and 4 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

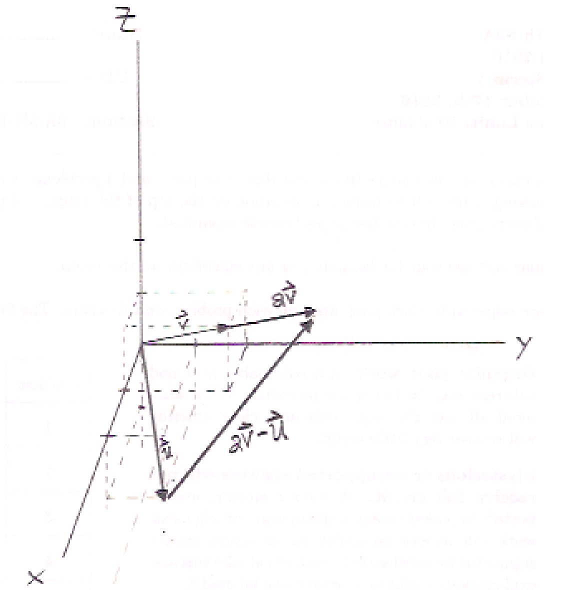
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- Write your solutions in the space below the questions. If you need more space, use the back of the page and clearly indicate when you have done this. Do not turn in your scratch paper.

Do not write in the table to the right.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

1. Let $\mathbf{v} = \langle 1, 2, 1\rangle$ and $\mathbf{u} = \langle 2, 1, -1\rangle$.

(a) (5 points) Draw the vector $2\mathbf{v} - \mathbf{u}$.



(b) (10 points) Find the parallel projection of $\mathbf{v}$ onto $\mathbf{u}$, i.e., find the vector $\mathbf{v}_{||\mathbf{u}}$.

We have

$$\begin{aligned}\mathbf{v}_{||\mathbf{u}} = \frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} &= \frac{\langle 1,2,1\rangle\cdot\langle 2,1,-1\rangle}{\langle 2,1,-1\rangle\cdot\langle 2,1,-1\rangle}\langle 2,1,-1\rangle \\ &= \frac{2+2-1}{4+1+1}\langle 2,1,-1\rangle \\ &= \frac{3}{6}\langle 2,1,-1\rangle \\ &= \frac{1}{2}\langle 2,1,-1\rangle = \langle 1, 1/2, -1/2\rangle.\end{aligned}$$

(c) (10 points) Find the area of the parallelogram spanned by $\mathbf{v}$ and $\mathbf{u}$.

We have

$$\mathbf{v}\times\mathbf{u} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 2 & 1\\ 2 & 1 & -1\end{vmatrix} = \begin{vmatrix}2 & 1\\ 1 & -1\end{vmatrix}\mathbf{i} - \begin{vmatrix}1 & 1\\ 2 & -1\end{vmatrix}\mathbf{j} + \begin{vmatrix}1 & 2\\ 2 & 1\end{vmatrix}\mathbf{k} = -3\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$$

or, equivalently, $\mathbf{v}\times\mathbf{u} = \langle -3, 3, -3\rangle$. As was shown in class, the area of the parallelogram $\mathcal{P}$ spanned by $\mathbf{u}$ and $\mathbf{v}$ is

$$\text{Area}(\mathcal{P}) = \|\mathbf{v}\times\mathbf{u}\| = \|\langle -3,3,-3\rangle\| = \sqrt{(-3)^2 + 3^2 + (-3)^2} = 3\sqrt{3}.$$

2. Given the points $P = (1, 2, 3)$, $Q = (3, 4, 4)$ and $R = (2, 2, 4)$, find:

(a) (5 points) The angle between $\overrightarrow{PQ}$ and $\overrightarrow{PR}$

Observe that $\overrightarrow{PQ} = \langle 3-1, 4-2, 4-3\rangle = \langle 2, 2, 1\rangle$ and $\overrightarrow{PR} = \langle 2-1, 2-2, 4-3\rangle = \langle 1, 0, 1\rangle$. Consequently, $\|\overrightarrow{PQ}\| = 3$, $\|\overrightarrow{PR}\| = \sqrt{2}$ and $\overrightarrow{PQ}\cdot\overrightarrow{PR} = 3$. Invoking the formula $\overrightarrow{PQ}\cdot\overrightarrow{PR} = \|\overrightarrow{PQ}\|\|\overrightarrow{PR}\|\cos\theta$ where θ is the angle between $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, we obtain

$$\cos\theta = \frac{\overrightarrow{PQ}\cdot\overrightarrow{PR}}{\|\overrightarrow{PQ}\|\|\overrightarrow{PR}\|} = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

Thus $\theta = \pi/4$ or $45°$.

(b) (10 points) A unit vector perpendicular to the plane containing P, Q and R

We know that $\mathbf{N} := \overrightarrow{PQ}\times\overrightarrow{PR}$ is perpendicular to the plane containing P, Q and R and so we compute it:

$$\mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 1 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 2 \\ 1 & 0 \end{vmatrix}\mathbf{k} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k} = \langle 2, -1, -2\rangle.$$

Thus, a unit vector perpendicular to the plane is found by normalizing $\mathbf{N}$. This is

$$\mathbf{n} = \frac{\mathbf{N}}{\|\mathbf{N}\|} = \frac{\langle 2, -1, -2\rangle}{\sqrt{2^2 + (-1)^2 + (-2)^2}} = \frac{1}{3}\langle 2, -1, -2\rangle = \langle 2/3, -1/3, -2/3\rangle.$$

(c) (5 points) The equation of the plane containing P, Q and R

We know that the plane containing P, Q and R is the set of points (x, y, z) such that $\langle x-1, y-2, z-3\rangle$ is perpendicular to $\mathbf{N}$ (or equivalently $\mathbf{n}$). Here we have used the base point $P = (1, 2, 3)$ but using Q and R will also yield the same result. Thus

$$0 = \mathbf{N}\cdot\langle x-1, y-2, z-3\rangle = 2(x-1) - 1(y-2) - 2(z-3) = 2x - y - 2z + 6$$

or equivalently

$$2x - y + 2z = -6.$$

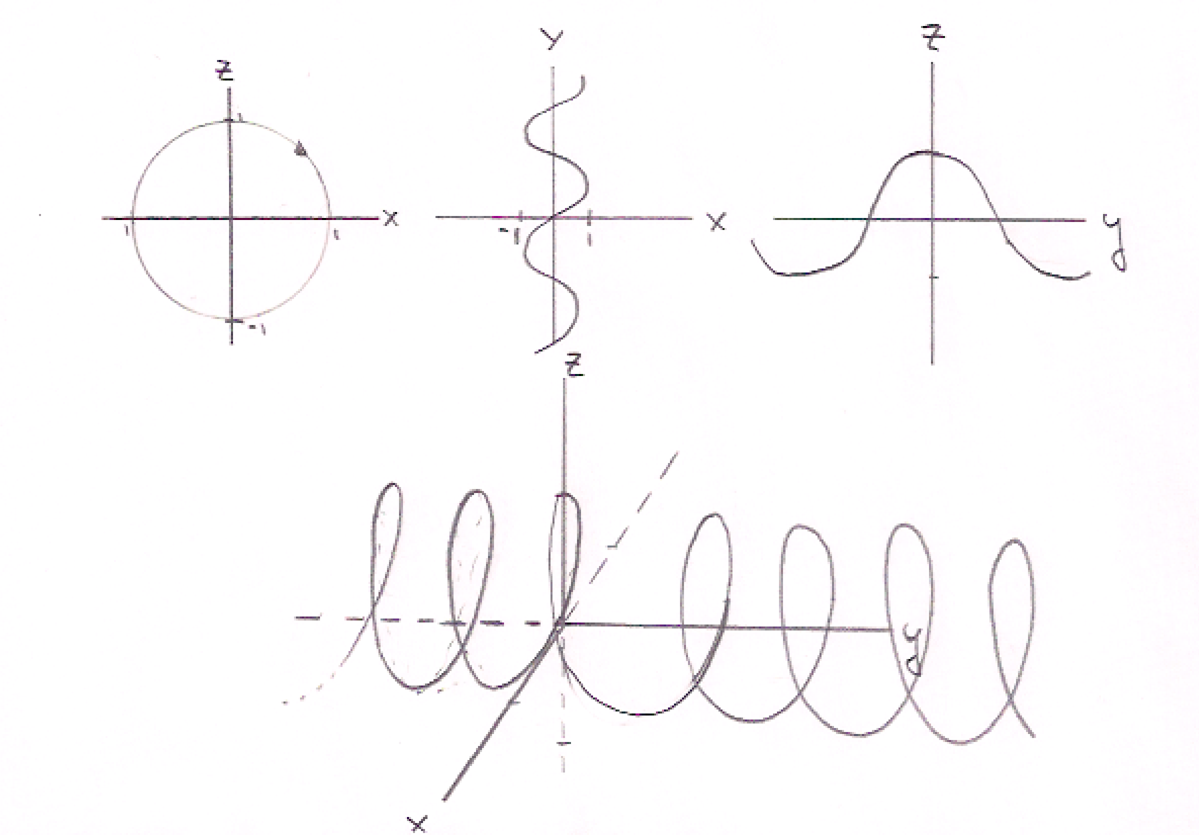
(d) (5 points) The distance from the plane containing P, Q and R to the point $S = (0, 1, 0)$

The distance d from the plane to the point S is, as we discussed in class, found by the parallel component of $\overrightarrow{PS}$ onto the unit vector $\mathbf{n}$. Thus,

$$\begin{aligned} d &= |\overrightarrow{PS}\cdot\mathbf{n}| = |\langle 0-1, 1-2, 0-3\rangle\cdot\mathbf{n}| \\ &= |\langle -1, -1, -3\rangle\cdot\langle 2/3, -1/3, -2/3\rangle| = 5/3. \end{aligned}$$

3. Consider the vector-valued function $\vec{\mathbf{r}}(t) = \langle \sin(t), t, \cos(t) \rangle$ for $-\infty < t < \infty$.

   (a) (10 points) Draw the projections of $\vec{\mathbf{r}}(t)$ to the three coordinate planes. Using these, sketch the curve determined by $\vec{\mathbf{r}}(t)$.



   (b) (15 points) Find a vector parameterization of the tangent line to $\vec{\mathbf{r}}(t)$ at $t = \pi/6$.
   We first compute $\vec{\mathbf{r}'}(t) = \langle \cos t, 1, -\sin t \rangle$. Thus, at $t = \pi/6$,

$$\vec{\mathbf{r}}(\pi/6) = \langle 1/2, \pi/6, \sqrt{3}/2 \rangle \text{ and } \vec{\mathbf{r}'}(\pi/6) = \langle \sqrt{3}/2, 1, -1/2 \rangle.$$

   Consequently, the line tangent to $\vec{\mathbf{r}}(t)$ at $t = \pi/6$ is given by

$$\begin{aligned} \vec{\ell}(s) &= s\left(\vec{\mathbf{r}'}(\pi/6)\right) + \vec{\mathbf{r}}(\pi/6) \\ &= s\langle \sqrt{3}/2, 1, -1/2 \rangle + \langle 1/2, \pi/6, \sqrt{3}/2 \rangle \\ &= \langle (\sqrt{3}s + 1)/2, s + \pi/6, (\sqrt{3} - s)/2 \rangle \end{aligned}$$

   defined for $-\infty < t < \infty$.

4. In answering the following question, recall that the zero vector is, by convention, orthogonal to every vector.

(a) (15 points) TRUE OR FALSE (circle one)

The dot product between two vectors is a scalar.	**[TRUE]**	FALSE
This follows directly from the definition of dot product		
The cross product between two vectors is a scalar.	TRUE	**[FALSE]**
This follows directly from the definition of cross product		
Two vectors $\mathbf{v}$ and $\mathbf{u}$ are orthogonal if and only if $\mathbf{v} \cdot \mathbf{u} = 0$.	**[TRUE]**	FALSE
This follows dirctly from the formula $\mathbf{v} \cdot \mathbf{u} = \|\mathbf{v}\|\|\mathbf{u}\| \cos\theta$.		
There exists a vector $\mathbf{v}$ such that $\mathbf{v} \times \langle 1,1,1\rangle = \langle 1,2,0\rangle$.	TRUE	**[FALSE]**
Recall that $\mathbf{v} \times \langle 1,1,1\rangle$ must be orthogonal to $\langle 1,1,1\rangle$.		
For any three vectors $\mathbf{v}, \mathbf{u}$ and $\mathbf{w}$, $\mathbf{v} \times (\mathbf{u} \times \mathbf{w}) = (\mathbf{v} \times \mathbf{u}) \times \mathbf{w}$.	TRUE	**[FALSE]**
Observe that $\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{k} = -\mathbf{j} \neq \mathbf{0} = (\mathbf{i} \times \mathbf{i}) \times \mathbf{j}$.		

(b) (10 points) Let $\vec{\mathbf{r}}(t)$ be differentiable and let C be a constant. Show that the following statement is true. Your reasoning/justification should be well-written and clear.

If $\|\vec{\mathbf{r}}(t)\| = C$ for all t, then $\frac{d}{dt}\vec{\mathbf{r}}(t) = \vec{\mathbf{r}'}(t)$ is orthogonal to $\vec{\mathbf{r}}(t)$ for all t.

Solution. Observe that

$$\vec{\mathbf{r}}(t) \cdot \vec{\mathbf{r}}(t) = \|\vec{\mathbf{r}}(t)\|^2 = C^2$$

for all t. Because $\vec{\mathbf{r}}$ is differentiable, we can differentiate both sides of the above equation to obtain

$$\frac{d}{dt}(\vec{\mathbf{r}}(t) \cdot \vec{\mathbf{r}}(t)) = \frac{d}{dt}C^2 = 0$$

for all t. By the product rule for the dot product, we have

$$2\left(\vec{\mathbf{r}'}(t) \cdot \vec{\mathbf{r}}(t)\right) = \vec{\mathbf{r}'}(t) \cdot \vec{\mathbf{r}}(t) + \vec{\mathbf{r}}(t) \cdot \vec{\mathbf{r}'}(t) = \frac{d}{dt}(\vec{\mathbf{r}}(t) \cdot \vec{\mathbf{r}}(t)) = 0$$

and so

$$\vec{\mathbf{r}'}(t) \cdot \vec{\mathbf{r}}(t) = 0$$

for all t. Therefore, the vectors $\vec{\mathbf{r}'}(t)$ and $\vec{\mathbf{r}}(t)$ are orthogonal for all t. □