20W-MATH32A-4 Midterm 2

HELEN WANG

TOTAL POINTS

93 / 100

QUESTION 1

Graph Matching 10 pts

1.1 a **0 / 2**

- **- 0 pts** Correct
- ✓ **- 2 pts Incorrect**

it appears that you correctly identified the periodicity (sin is periodic), but did not account for the subtraction x-y. Here is an idea: what is the set of x and y so that x-y=c?

1.2 b **2 / 2**

- ✓ **- 0 pts Correct**
- **- 2 pts** Incorrect

1.3 c **2 / 2**

- ✓ **- 0 pts Correct**
- **- 2 pts** Incorrect

1.4 d **2 / 2**

- ✓ **- 0 pts Correct**
- **- 2 pts** Incorrect

1.5 e **2 / 2**

- ✓ **- 0 pts Correct**
- **- 2 pts** Incorrect

QUESTION 2

Limits 15 pts

2.1 Ratio Cancel **5 / 5**

- ✓ **+ 3 pts Factor**
- ✓ **+ 2 pts Answer**
- **+ 3 pts** Alternate method
- **+ 0 pts** No useful progress

2.2 Limit Along 2 directions **5 / 5**

- ✓ **+ 3 pts Polar/use lines of arbitrary slope/etc**
- ✓ **+ 2 pts Answer**
- **+ 0 pts** No useful progress
- **+ 2 pts** Partial credit

2.3 Limit Polar **5 / 5**

- ✓ **+ 3 pts Polar**
- ✓ **+ 2 pts Answer**
- **+ 0 pts** No useful progress
- **+ 1 pts** Partial credit

QUESTION 3

Linearization 30 pts

3.1 Continiity **5 / 5**

- ✓ **- 0 pts Correct**
- **- 5 pts** No work shown

3.2 Partial X **8 / 8**

- ✓ **- 0 pts Correct**
- **- 3 pts** Incorrect partial deriavtive
- **- 2 pts** Incorrect description of the domain of f_x
- **- 8 pts** No work shown

3.3 Partial Y **6 / 8**

- **- 0 pts** Correct
- **- 2 pts** Incorrect partial derivative
- ✓ **- 2 pts Incorrect description of the domain**
- **- 8 pts** No work shown

The domain of the derivative cannot be larger than that of the initial function.

3.4 Differentiability **4 / 4**

- ✓ **- 0 pts Correct**

- **4 pts** No work shown
- **2 pts** -2

3.5 The linearization 5 / 5

✓ **- 0 pts Correct**

- 1 pts Evaluation of partial derivatives

- 5 pts No work shown

QUESTION 4

Frenet Frame 30 pts

4.1 Speed 3 / 5

✓ **+ 2 pts Compute r’**

+ 3 pts Answer

✓ **+ 1 pts Partial credit**

+ 0 pts No useful progress

4.2 Tangent Vector 5 / 5

✓ **+ 2 pts Definition**

✓ **+ 3 pts Answer**

+ 0 pts No useful progress

+ 1 pts Partial credit

Ok given (a)

4.3 Acceleration 5 / 5

✓ **+ 3 pts look at r’’**

✓ **+ 2 pts Answer**

+ 0 pts No useful progress

+ 1 pts Partial credit

4.4 Normal and Curvature 5 / 5

✓ **+ 3 pts Use accel dot T or equivalent**

✓ **+ 2 pts Answer**

+ 0 pts No useful progress

+ 3 pts Other valid method

+ 1 pts Partial credit

4.5 Radius of Osculating Circle 3 / 3

✓ **+ 2 pts Curvature**

✓ **+ 1 pts Radius is 1/curvature**

+ 0 pts No useful progress

+ 1 pts Partial credit

Ok given answers above

4.6 Normal Vector 1 / 2

+ 2 pts Answer

+ 0 pts No useful progress

✓ **+ 1 pts Partial credit**

r’ does not have constant norm so T’ is not this easy to compute directly

4.7 Frenet Frame 5 / 5

✓ **+ 2 pts Cross product**

✓ **+ 3 pts Answer**

+ 0 pts No useful progress

+ 1 pts Partial credit

Ok given previous

QUESTION 5

Limits 15 pts

5.1 First estimate 5 / 5

✓ **- 0 pts Correct**

- 1 pts Careful with that distribution! Otherwise, right idea.

- 1 pts Requires bit more justification

Non-negative

5.2 Squeeze Theorem 5 / 5

✓ **- 0 pts Correct**

- 5 pts No work shown

- 1 pts Requires additional justification

5.3 5 / 5

✓ **- 0 pts Correct**

- 1 pts What about points (x,y) which are not (0,0)?

- 5 pts No work shown

Midterm 2

Last Name: Wang

First Name: Helen

Student ID: 405370396

Signature: HWang

Section:

Tuesday: 1A

Thursday: 1B

TA: Bertrand Stone

Instructions: Do not open this exam until instructed to do so. You will have 50 minutes to complete the exam. Please print your name and student ID number above, and circle the number of your discussion section. **You may not use a graphing calculators,** books, phones, notes, or any other material to help you. Please make sure **your phone is silenced** and stowed where you cannot see it. You may use any available space on the exam for scratch work. If you need more scratch paper, please ask one of the proctors. You must **show your work** to receive credit. Please circle or box your final answers.

Please do not write below this line.

Question	Points	Score
1	5	
2	15	
3	30	
4	30	
5	20	
Total:	100	

1. (5 points) Match each equation with its contour plot.

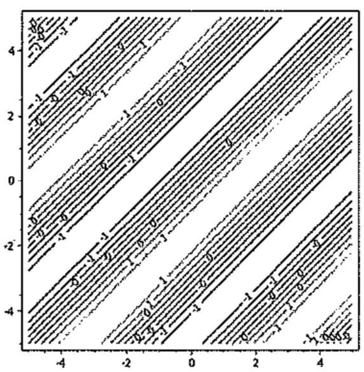
(i) $f(x, y) = \sin(x - y)$. b

(ii) $f(x, y) = x^3 - 8y$. d

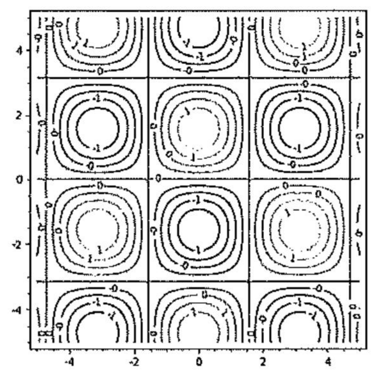
(iii) $f(x, y) = |x| + |y|$. c

(iv) $f(x, y) = \sin(x)\sin(y)$. b

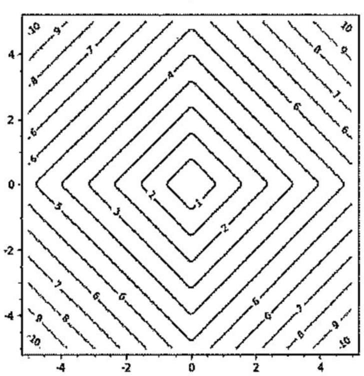
(v) $f(x, y) = y^2 - 4x^2$. e



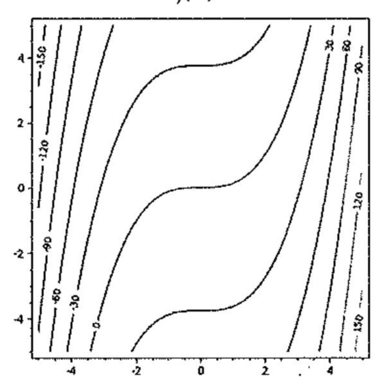
(a)



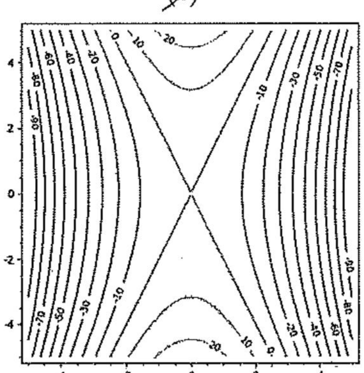
(b)



(c)



(d)



(e)

2. Compute the following limits or show they do not exist:

(a) (5 points) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4 - y^4}{x^2 + y^2}$

$$\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2} = \lim \frac{(x^2-y^2)\cancel{(x^2+y^2)}}{\cancel{(x^2+y^2)}}$$

$$= \lim_{(x,y)\to(0,0)} x^2 - y^2 = 0$$

$$y = mx \Rightarrow \lim_{(x,y)\to(0,0)} x^2 - (mx)^2 -$$

(b) (5 points) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 + 2y^2}{x^2 + y^2}$

$$\lim_{(x,y)\to(0,0)} \frac{x^2+2y^2}{x^2+y^2}$$

$$y = mx \quad \Rightarrow \lim_{(x,y)\to(0,0)} \frac{x^2+2(mx)^2}{x^2+(mx)^2}$$

$$= \lim \frac{x^2+2m^2x^2}{x^2+m^2x^2} = \lim \frac{\cancel{x^2}(1+2m^2)}{\cancel{x^2}(1+m^2)}$$

$$= \lim_{(x,y)\to(0,0)} \frac{1+2m^2}{1+m^2}$$

limit DNE because it is dependent on m, which varies for different paths to the point.

Question 2 continues on the next page...

(c) (5 points) $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$. Hint: $x = r\cos(\theta)$...

$x = r\cos\theta$
$y = r\sin\theta$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{r\cos\theta\, r\sin\theta}{\sqrt{(r\cos\theta)^2+(\sin\theta)^2}}$$

$$= \lim_{(x,y)\to(0,0)} \frac{r^2\cos\theta\sin\theta}{\sqrt{r^2(\cos^2\theta+\cancel{\sin^2\theta})}} = \lim \frac{r^2\cos\theta\sin\theta}{\cancel{\sqrt{r^2}}\, r}$$

$$= \lim_{r\to 0} r\cos\theta\sin\theta = 0$$

3. For this question, we will focus on $f(x, y) = \ln(x^2 + y^2 - 4)$.

(a) (5 points) Where is the function f continuous? Hint: where is this defined?

$f(x,y)$ = continuous if $f(a,b) = \lim_{(x,y)\to(a,b)} f(x,y)$ for all points on domain

$\ln(x^2+y^2-4)$ ← defined when this part isn't negative or 0

$\Rightarrow x^2+y^2-4 > 0$

$x^2+y^2 > 4$

x^2 min $= 0 \Rightarrow y \geq 2$ domain: $y \Rightarrow y > \sqrt{4-x^2}$, $y < -\sqrt{4-x^2}$

y^2 min $= 0 \Rightarrow x \geq 2$ $x \Rightarrow x > \sqrt{4-y^2}$, $x < -\sqrt{4-y^2}$

Function = defined for $x^2+y^2 > 4$

$\ln(x)$
$= \log_e x$
$= e^? = x$
x can't be
≤ 0

(b) (8 points) Compute f_x, the partial with respect to x and determine where it is continuous. Hint: the domain of f_x cannot be larger than that of f.

$f_x = \frac{\partial f}{\partial x}(\ln(x^2+y^2-4)) = \frac{1}{x^2+y^2-4} = \frac{2x}{x^2+y^2-4}$

Domain: f_x = cont. when $(x^2+y^2-4) \neq 0$

$x^2+y^2-4 \neq 0$

$x^2+y^2 \neq 4$

$y^2 = \sqrt{4-x^2}$

$y \neq +\sqrt{4-x^2}, -\sqrt{4-x^2}$

$x \neq +\sqrt{4-y^2}, -\sqrt{4-y^2}$

$2x$, x^2+y^2-4 = 2 continuous functions, composite = cont.

(c) (8 points) Compute f_y, the partial with respect to y and determine where it is continuous.

$f_y = \frac{\partial f}{\partial y}(\ln(x^2+y^2-4)) = \frac{1}{x^2+y^2-4} \cdot 2y = \frac{2y}{x^2+y^2-4}$

Domain: f_y = cont. when $x^2+y^2-4 \neq 0$

same domain as f_x

Question 3 continues on the next page...

Question 3 continued...

(d) (4 points) Explain why f is differentiable on its domain.

f = diff on domain if f = continuous and limit for $f(a,b)$ equal defined point at $f(a,b)$. Since all points in the domain are defined, and the function and partial derivatives are continuous on the domain, f is diff.

$f(x,y)$ = diff. @ (a,b) if ~~f(x,y)~~ $f_x(a,b)$ && $f_y(a,b)$ exist,

and $\lim\limits_{(x,y)\to(a,b)} \dfrac{f(x,y) - L(x,y)}{\sqrt{(x-a)^2 + (y-b)^2}} = 0$

(e) (5 points) Compute the linearization of the function at the point $(1, 2, f(1, 2))$.

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$$f(x,y) = \ln(x^2+y^2-4)$$

$$f_x = \frac{2x}{x^2+y^2-4}, \quad f_y = \frac{2y}{x^2+y^2-4}$$

$$L(x,y) = f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2)$$

$$= \ln(1^2+2^2-4) + \frac{2}{1+4-4}(x-1) + \frac{2(2)}{1+4-4}(y-2)$$

$$= \ln(1) + 2(x-1) + 4(y-2)$$

$$= 2(x-1) + 4(y-2)$$

4. Here, we'll consider a particle moving in a spiral $\vec{r}(t) = \langle t, t\sin(t), t\cos(t)\rangle$.

(a) (5 points) What is the speed of the particle at the point $(0, 0, 0)$?

speed $= |v(t)| = |r'(t)|$

$r'(t) = \langle 0,\ t\cos t + 1\sin t,\ -t\sin t + 1\cos t\rangle$

$= \langle 0,\ t\cos t + \sin t,\ \cos t - t\sin t\rangle \Big|_{\langle 0,0,0\rangle}$

@ $\langle 0,0,0\rangle$, $t = 0$

speed $= |r'(0)| = \sqrt{0^2 + (0 + \sin 0)^2 + (\cos 0 - 0)^2}$

$= \sqrt{0 + 0 + 1^2} = 1$

(b) (5 points) What is the unit tangent vector $\vec{T}$ of this particle at the point $(0, 0, 0)$?

$T = \dfrac{r'(t)}{\|r'(t)\|} = \dfrac{v(t)}{\|v(t)\|} = \dfrac{\langle 0,\ t\cos t + \sin t,\ \cos t - t\sin t\rangle}{1}$

$= \dfrac{\langle 0,\ 0 + 0,\ 1 - 0\rangle}{1} = \langle 0, 0, 1\rangle$

Question 4 continues on the next page. . .

Question 4 continued...

(c) (5 points) Compute the acceleration of this particle at the point $(0, 0, 0)$.

$a_T = a_T T + a_N N = v'(t) = r''(t)$

$v(t) = \langle 0, t\cos t + \sin t, \cos t - t\sin t \rangle$

$a_T = v'(t) = \langle 0, -t\sin t + \cos t + \cos t, -\sin t - (t\cos t + \sin t) \rangle$

$= \langle 0, 2\cos t - t\sin t, t\cos t - 2\sin t \rangle \quad (0,0,0)$

$t = 0$

$= \langle 0, 2\cos 0 - 0, 0 - \cancel{2\sin 0}^{0} \rangle$

$= \langle 0, 2, 0 \rangle$

$|a_T| = \sqrt{0 + 4 + 0} = 2$

(d) (5 points) What is the normal component of the acceleration at the point $(0, 0, 0)$?

$a = a_T T + a_N N$

$a_N N = a - a_T T$

$= 2 - \langle 0, 2, 0 \rangle \cdot \langle 0, 0, 1 \rangle$

$= 2 - (0 + 0 + 0)$

$= 2$

(e) (3 points) What is the radius of the osculating circle at the point $(0, 0, 0)$?

radius of osci circle points to cardio

$a = \dfrac{v(t)^2}{r}$

$r = \dfrac{r'(t)^2}{a} = \dfrac{|v(t)|^2}{a} = \dfrac{1^2}{2} = \dfrac{1}{2}$

Question 4 continues on the next page...

(f) (2 points) Give the unit normal vector $\vec{\mathbf{N}}$ at the point $(0,0,0)$.

$N(t) = \dfrac{T'(t)}{\|T'(t)\|}$ ~~$\langle 0, t\sin t + \cos t \sin t$~~

$T(t) = \langle 0, t\cos t + t\sin t, \cos t - t\sin t\rangle$

$N(t) = \dfrac{T'(t)}{\|T'(t)\|} = \langle 0, (-t\sin t + t\cos t) + \cos t, \sin t - (t\cos t + t\sin t)\rangle$

$= \langle 0, 2\cos t - t\sin t, 2\sin t - t\cos t\rangle$

$= \dfrac{\langle 0, 2(1) - 0, 0 - 0\rangle}{\|T'(t)\|} = \dfrac{\langle 0, 2, 0\rangle}{\sqrt{4}}$

$= \langle \frac{0}{2}, \frac{2}{2}, \frac{0}{2}\rangle = \langle 0, 1, 0\rangle$

(g) (5 points) Compute the Frenet frame at the point $(0,0,0)$.

Frenet frame $= \{T, N, B\}$

$T =$ already found to be $\langle 0, 0, 1\rangle$

$N =$ already found to be $\langle 0, 1, 0\rangle$

$B = T \times N$

$$\begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = i(0-1) + j(0) + k(0)$$

$= B = \langle -1, 0, 0\rangle$

5. (a) (5 points) Show that for any numbers, a, b, we have

$$(|a| + |b|)^4 \geq a^4 + b^4.$$

Hint: start by distributing the left hand side.

$(|a| + |b|)^4 \geq a^4 + b^4$

$(|a| + |b|)^2 (|a| + |b|)^2$

$= (|a|^2 + 2|a||b| + |b|^2)^2$

$= (a^2 + b^2 + \underbrace{2|ab|}_{\text{all terms} \geq \text{positive}})^2$

$= a^4 + b^4 + \underbrace{2|ab|a^2 + b^2|2ab| + \ldots}_{\text{positive}} \geq a^4 + b^4$ ✓

$(|a| + |b|)^4$ is at least $a^4 + b^4$, so it is always $\geq a^4 + b^4$,

(b) (10 points) Use the previous fact to show that

$$\lim_{(x,y)\to(0,0)} \frac{x^4 + y^4}{|x| + |y|} = 0$$

$\lim_{(x,y)\to(0,0)} \frac{x^4+y^4}{|x|+|y|} = 0$

$\frac{a^4+b^4}{|a|+|b|} \leq \frac{(|a|+|b|)^4}{|a+b|}$

$0 \leq \frac{x^4+y^4}{|x|+|y|} \leq (|x|+|y|)^3$

$\lim_{(x,y)\to(0,0)} (|x|+|y|)^3 = (0+0)^3 = 0$

by squeeze thm, since $0 \leq \frac{x^4+y^4}{|x|+|y|} \leq (|x|+|y|)^3$

and $\lim_{(x,y)\to(0,0)} 0 = 0$, $\lim_{(x,y)\to(0,0)} (|x|+|y|)^3 = 0$,

~~the~~ limit of $\lim_{(x,y)\to(0,0)} \frac{x^4+y^4}{|x|+|y|} = 0$, too.

Question 5 continues on the next page...

(c) (5 points) Explain the why the function

$$f(x,y) = \begin{cases} \dfrac{x^4+y^4}{|x|+|y|} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

is continuous on its domain. Make sure you explain why it is continuous at every point, not just $(0,0)$.

$f(x,y) = x^4+y^4$ is continuous on the domain and $f(x,y) = |x|+|y|$ is continuous on the domain, so the composite of $\frac{x^4+y^4}{|x|+|y|}$ is continuous on the domain except for when $(x,y) = (0,0)$.

$f(x,y)$ is continuous if every pt on the domain is defined, has a limit and for every pt (a,b) $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

For $(x,y) = (0,0)$

$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{x^4+y^4}{|x|+|y|} = 0$, as shown in the previous part B, so $f(x,y)$ is continuous @ $(x,y) = (0,0)$.

altogether, the function is continuous on the entire domain.