20W-MATH32A-4 Midterm 1

HELEN WANG

TOTAL POINTS

98 / 100

QUESTION 1

1 True/False 5 / 5

- **- 1 pts** Part A
- **- 1 pts** Part B
- **- 1 pts** Part C
- **- 1 pts** Part D
- **- 1 pts** Part D
- ✓ **- 0 pts Perfect**

Don't worry, I checked this against your original paper.

QUESTION 2

Plane 25 pts

2.1 Equation of Plane 15 / 15

- ✓ **+ 10 pts Obtain normal**
- ✓ **+ 5 pts Equation of plane**
- **+ 0 pts** No useful progress
- **+ 3 pts** Wrong constant/sign error
- **+ 8 pts** Error in normal computation
- **+ 2 pts** Partial credit
- **+ 5 pts** Partial credit for normal

2.2 Area of triange 5 / 5

- ✓ **+ 3 pts Using normal/cross product**
- ✓ **+ 2 pts Answer**
- **+ 0 pts** No useful progress
- **+ 3 pts** Other method
- **+ 1 pts** Partial credit

2.3 Coplanar 5 / 5

- ✓ **+ 3 pts Check in equation**
- ✓ **+ 2 pts Answer**
- **+ 0 pts** No useful progress
- **+ 3 pts** Other method
- **+ 2 pts** Partial credit

QUESTION 3

3 Line/Plane Intersection 20 / 20

- ✓ **+ 7 pts Describe line**
- ✓ **+ 3 pts Check plane**
- ✓ **+ 5 pts Solve, t = 2**
- ✓ **+ 5 pts Conclude, find (5, -2, 3)**
- **+ 0 pts** No useful progress
- **+ 4 pts** Partial credit for line
- **+ 3 pts** Misc partial credit

QUESTION 4

Projections/Minimum Distance 25 pts

4.1 Compute Length of Projection 10 / 10

- ✓ **- 0 pts Correct**
- **- 0 pts** Identifying the normal vector
- **- 0 pts** Finding a point
- **- 0 pts** Computing the vector RP
- **- 0 pts** Computing the projection
- **- 0 pts** Computing the norm
- **- 3 pts** Critical computation error
- **- 2 pts** Incorrect length
- **- 10 pts** No work shown

4.2 Closest Point 10 / 10

- ✓ **- 0 pts Correct**
- **- 10 pts** No work given
- **- 3 pts** Critical calculation mistake

4.3 Closest point on Sphere 5 / 5

- ✓ **- 0 pts Correct**
- **- 5 pts** No work shown
- **- 2 pts** Wrong direction for normal vector!
- **- 3 pts** Critical computation error

- **1 pts** Arithmetic error

QUESTION 5

Intersection of Paths and Angles 25 pts

5.1 Sketch 2d Curve **8 / 10**

- **0 pts** Correct

✓ - **2 pts** **No Labels**

- **4 pts** Incorrect graph

- **10 pts** No work shown

- **2 pts** Incorrect range

- **1 pts** Incorrect direction

5.2 Compute Intersection **10 / 10**

✓ - **0 pts** **Correct**

- **2 pts** Missing Point of Intersection

- **3 pts** Not checking for intersection

- **3 pts** Incorrect substitution (using t=1 instead of finding value of t)

- **4 pts** No work shown

- **10 pts** No work given

- **3 pts** Critical arithmetic error

5.3 Compute Angle **5 / 5**

✓ - **0 pts** **Correct**

- **3 pts** Incorrect dot product computation

- **2 pts** Incorrect angle

- **0 pts** Error from previous part

- **1 pts** Incorrect differentiation

- **2 pts** Incorrect angle computation

- **1 pts** Computation error

- **10 pts** No work shown

Midterm 1

Last Name: Wang

First Name: Helen

Student ID: 405320396

Signature: Hwang

Section: Tuesday: 4A Thursday: 4B TA: Bertrand Stone

Instructions: Do not open this exam until instructed to do so. You will have 50 minutes to complete the exam. Please print your name and student ID number above, and circle the number of your discussion section. **You may not use a graphing calculators,** books, phones, notes, or any other material to help you. Please make sure **your phone is silenced** and stowed where you cannot see it. You may use any available space on the exam for scratch work. If you need more scratch paper, please ask one of the proctors. You must **show your work** to receive credit. Please circle or box your final answers.

Please do not write below this line.

Question	Points	Score
1	5	
2	25	
3	20	
4	25	
5	25	
Total:	100	

1. True or False:

(a) (1 point) For any vector $\vec{v}$ in $\mathbb{R}^3$,

$\vec{v} \cdot \vec{v} = \|v\|^2$

$$\|\vec{v} \times \vec{v}\| = \|\vec{v}\|^2 .$$ False

(b) (1 point) If $\|\vec{v}\| = \|\vec{w}\|$, then $\vec{v} = \pm\vec{w}$ False

(c) (1 point) For any pair of vectors $\vec{u}, \vec{v}$ in $\mathbb{R}^3$, we have that True

$$(\underbrace{\vec{u} \times \vec{v}}_{\perp \vec{u}}) \cdot \vec{u} = \vec{0}.$$

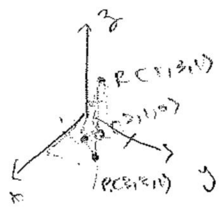
(d) (1 point) If $\vec{v}$ and $\vec{w}$ are parallel vectors, then they have the same di-

(e) (1 point) If $\vec{v}$ is a direction vector for a line, then so are 2\

Parametric
· line through
param =
· circle w/ rad
· cycloid by
· slope
· area
· vector-val
· path =
· $\frac{d}{dt}\vec{r}(t) =$
· dot
· cross
· intersecting plane
· check if 2 lines intersect
- $r'(t_0)$ points tangent to path $r(t_0)$
= tangent vector
· tangent line
· if $R_1(t) = R_2(t)$

2. Consider the points $P = (3, 3, 1)$, $Q = (2, 1, 0)$, and $R = (1, 3, 1)$.

(a) (15 points) Find the equation of the plane containing the points P, Q, and R.



Plane PQR : $P = (3,3,1)$, $Q = (2,1,0)$, $R = (1,3,1)$

$\vec{PQ} = \langle 2-3, 1-3, 0-1 \rangle = \langle -1, -2, -1 \rangle$

$\vec{PR} = \langle 1-3, 3-3, 1-1 \rangle = \langle -2, 0, 0 \rangle$

$\vec{n} \perp (\vec{PQ}$ and $\vec{PR}) = \vec{PQ} \times \vec{PR}$

$\vec{PQ} \times \vec{PR} = \langle -1, -2, -1 \rangle \times \langle -2, 0, 0 \rangle$

$$= \begin{vmatrix} i & j & k \\ -1 & -2 & -1 \\ -2 & 0 & 0 \end{vmatrix} = i \begin{vmatrix} -2 & -1 \\ 0 & 0 \end{vmatrix} - j \begin{vmatrix} -1 & -1 \\ -2 & 0 \end{vmatrix} + k \begin{vmatrix} -1 & -2 \\ -2 & 0 \end{vmatrix}$$

$= i(0) - j(0+2) + k(0+4) = -2j + 4k = \langle 0, -2, 4 \rangle$

plane = $\vec{n}$ and a point on plane

Plane PQR $\Rightarrow$ $\vec{n} = \langle a, b, c \rangle$, point = $Q = (2,1,0)$

$= a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

$0(x-2) + (-2)(y-1) + 4(z-0) = 0$

Plane PQR $= -2(y-1) + 4z = 0$

Question 2 continues on the next page...

Question 2 continued...

(b) (5 points) Find the area of the triangle formed by the points P, Q, and R.

area of $\triangle PQR = \frac{1}{2}bh = \frac{1}{2}\|\vec{v} \times \vec{w}\| = \frac{1}{2}\|\vec{PQ} \times \vec{PR}\|$

$\vec{PQ} = \langle -1, -2, -1 \rangle$ $\qquad \vec{PQ} \times \vec{PR} = \langle 0, -2, 4 \rangle$

$\vec{PR} = \langle -2, 0, 0 \rangle$ $\qquad \|\vec{PQ} \times \vec{PR}\| = \sqrt{0^2 + (-2)^2 + 4^2} = \sqrt{4+16} = \sqrt{20}$

$\frac{1}{2}\|\vec{PQ} \times \vec{PR}\| = \frac{1}{2}(\sqrt{20}) = \frac{1}{2}(2\sqrt{5}) = \sqrt{5}$ units^2

(c) (5 points) Are the four points P, Q, R, and $S = (7, 4, 0)$ coplanar? Justify your answer.

4 points P, Q, R and S are coplanar if S is on plane PQR and is valid in plane equation for PQR $= -2(y-1) + 4z = 0$

$S = (7, 4, 0)$

$-2(4-1) + 4(0) = -2(3) + 0 = -6 \neq 0$

No because S isn't valid in the plane equation, so the 4 points are not on the same plane.

3. (20 points) Let L be the line through the points $(1, 0, 7)$ and $(3, -1, 5)$. Does L intersect the plane $x + y + z = 6$? If not, justify your answer. If L does intersect the plane, find the intersection.

L = line w/ point $P_0 = (1, 0, 7)$, $P_2 = (3, -1, 5)$

$L = (1, 0, 7) + t\langle 3-1, -1-0, 5-7\rangle.$

$= (1, 0, 7) + t\langle 2, -1, -2\rangle$

$= \langle \overset{x_L}{1+2t}, \overset{y_L}{-t}, \overset{z_L}{7-2t}\rangle$

plane P = $x + y + z = 6$

$(1+2t) + (-t) + (7-2t) = 6$

$1 + \cancel{2t} \overset{+t}{\cancel{-t}} + 7 - 2t = 6$

$8 - t = 6$

$8 - 6 = t \Rightarrow (t = 2)$

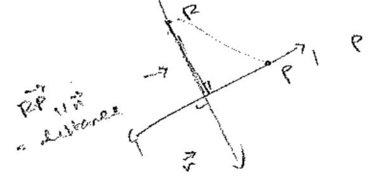
$L(2) = \langle 1+2(2), -2, 7-2(2)\rangle$

$= (1+4, -2, 7-4) = (5, -2, 3)$

Yes, L intersects the plane @ point $(5, -2, 3)$.

4. Fix a point P on a plane $\mathcal{P}$. Recall that the distance between a point R and a plane $\mathcal{P}$ in $\mathbb{R}^3$ with normal vector $\vec{n} = \langle a, b, c\rangle$ is given by

$$\left\|\vec{RP}_{\|\vec{n}}\right\|,$$



the length of the orthogonal projection of $\vec{RP}$ on to $\vec{n}$.

(a) (10 points) Compute the distance between the point $R = (4, -2, 2)$ and the plane with vector form by $\langle 2, -1, 2\rangle \cdot \langle x, y, z\rangle = -4$.

$R = (4,-2,2)$, plane: $2x - y + 2z = -4$ $\quad \|\vec{n}\| = \sqrt{2^2+1^2+2^2}$

$\vec{n} = \langle 2,-1,2\rangle$ $\quad = \sqrt{4+1+4}$

distance b/w R and plane $= \|\vec{RP}_{\|\vec{n}}\|$ $\quad = \sqrt{9} = 3$

P = point on the line $= (1,6,0)$

$z = 0$

$2x - y = -4$

$2(1)+4 = y$

$x = 1,\ y = 2+4 = 6$

$\vec{RP} = \langle 1-4, 6+2, 0-2\rangle = \langle -3, 8, -2\rangle$

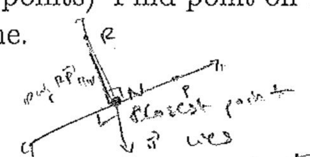
$\vec{RP}_{\|\vec{n}} = \left(\frac{u\cdot v}{\|v\|}\right)\left(\frac{v}{\|v\|}\right) = \left(\frac{\vec{RP}\cdot\vec{n}}{\|\vec{n}\|}\right)\left(\frac{\vec{n}}{\|\vec{n}\|}\right) = \left(\frac{\langle -3,8,-2\rangle \cdot \langle 2,-1,2\rangle}{3}\right)\left(\frac{\langle 2,-1,2\rangle}{3}\right)$

$= \frac{(-6)+(-8)+(-4)}{3}\left(\langle \frac{2}{3}, \frac{-1}{3}, \frac{2}{3}\rangle\right)$

$= \cancel{\frac{-18}{3}}\ -6\left(\langle \frac{2}{3}, \frac{-1}{3}, \frac{2}{3}\rangle\right) = \langle -\frac{12}{3}, +\frac{6}{3}, -\frac{12}{3}\rangle = \langle -4, +2, -4\rangle$

$\|\vec{RP}_{\|\vec{n}}\| = \sqrt{4^2+2^2+4^2} = \sqrt{16+4+16} = \sqrt{36} = 6$ units

(b) (10 points) Find point on $\mathcal{P}$ that is closest to R. Hint: $\vec{RP}_{\|\vec{n}}$ points "towards" the plane.



point N closest to R = 6 units away from R in dir of plane P, along the $\vec{RP}_{\|\vec{n}}$ ⊥ to plane P

$N = R + 6\left(\frac{\vec{RP}_{\|\vec{n}}}{\|\vec{RP}_{\|\vec{n}}\|}\right) = (4,-2,2) + 6\,\frac{\langle -4, +2, -4\rangle}{6}$

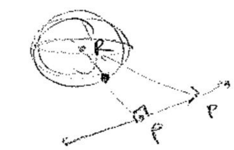
$= (4-4, -2+2, 2-4)$

$= (0, 0, -2)$

Question 4 continues on the next page...

Question 4 continued...

(c) (5 points) Which point on the sphere $(x-4)^2+(y+2)^2+(z-2)^2 = 16$ is closest to the plane $\mathcal{P}$?



sphere $S = (x-4)^2 + (y+2)^2 + (z-2)^2 = 16$

C = centre = $(4,-2,2)$, $R = 4$

point on sphere closest to plane P = point N

on the edge of sphere, R units away

from centre, in direction of $\vec{CP}_{\|\vec{n}}$ where $\vec{n}$ = normal vector to plane

$P = (1,6,0)$

$\vec{CP} = \langle 1-4, 6+2, 0-2\rangle$

$= \langle -3, 8, -2\rangle$

$\vec{CP}_{\|\vec{n}} = \langle -4, +2, -4\rangle$

$\|\vec{CP}_{\|\vec{n}}\| = 6$

$$N = C + R\left(\frac{\vec{CP}_{\|\vec{n}}}{\|\vec{CP}_{\|\vec{n}}\|}\right)$$

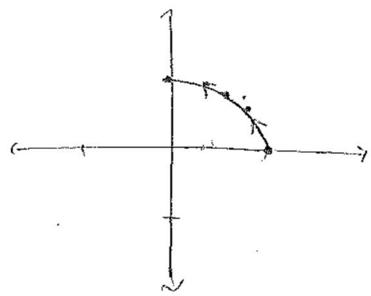
$$= (4,-2,2) + \cancel{4}^{\,2}\left(\frac{\langle -4,+2,-4\rangle}{\cancel{6}\;3}\right)$$

$$= (4,-2,2) + \langle -\tfrac{8}{3}, \tfrac{4}{3}, -\tfrac{8}{3}\rangle$$

$$= \left(4-\tfrac{8}{3},\ -2+\tfrac{4}{3},\ 2-\tfrac{8}{3}\right)$$

5. (a) (10 points) If a and b are positive numbers, sketch the curve in $\mathbb{R}^2$ traced out by $\langle a\cos(t), b\sin(t)\rangle$ when $0 \le t \le \frac{\pi}{2}$.

t	x/a	y/b
0	1	0
$\pi/6$	$\sqrt{3}/2$	$1/2$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$
$\pi/3$	$1/2$	$\sqrt{3}/2$
$\pi/2$	0	1



(b) (10 points) What point lies on the intersection of the curves traced out by the functions $\vec{r}(t) = \langle\cos(t), 2\sin(t), 1\rangle$ and $\vec{p}(s) = \langle 3s^3 - 9s + 6, 2e^{s-1}, 2s - 1\rangle$?

$\vec{r}(t) = \langle \cos(t), 2\sin(t), 1\rangle$

$\vec{p}(s) = \langle 3s^3 - 9s + 6, 2e^{s-1}, 2s-1\rangle$

$\cos(t) = 3s^3 - 9s + 6 = 3(1) - 9 + 6 = 3 - 9 + 6 = 0 \quad (t = \pi/2)$

$2\sin(t) = 2e^{s-1} = 2e^{1-1} = 2e^0 = 2, \quad \sin(t) = 1 \quad (t = \pi/2)$

$1 = 2s - 1 \Rightarrow 2 = 2s \Rightarrow s = 2/2 = 1$

$s = 1, \quad t = \pi/2$

point $= (\cos(\pi/2), 2\sin(\pi/2), 1)$

$= (0, 2, 1)$

Question 5 continues on the next page...

Question 5 continued...

(c) (5 points) The angle of intersection between two curves is the angle between their tangent vectors at the point of intersection. Compute $\vec{r}'$ and $\vec{p}'$, then find the angle of intersection.

$\vec{r}(t) = \langle \cos(t), 2\sin(t), 1 \rangle$

$\vec{r}'(\pi/2) = \langle -\sin(t), 2\cos(t), 0 \rangle = \langle -\sin(\pi/2), 2\cos(\pi/2), 0 \rangle$

$= \langle -1, 0, 0 \rangle$ $\qquad \|\vec{r}'(\pi/2)\| = \sqrt{1^2 + 0^2 + 0^2} = 1$

$\vec{p}(t) = \langle 3s^3 - 9s + 6, 2e^{s-1}, 2s - 1 \rangle$

$\vec{p}'(s) = \langle 9s^2 - 9, 2e^{s-1}, 2 \rangle$

$\vec{p}'(1) = \langle 9 - 9, 2e^0, 2 \rangle = \langle 0, 2, 2 \rangle$ $\qquad \|\vec{p}'(t)\| = \sqrt{0^2 + 4 + 4} = 2\sqrt{2}$

θ of intersection $\Rightarrow$ $\vec{v} \cdot \vec{w} = \|v\| \|w\| \cos\theta$

$\theta = \cos^{-1}\left(\frac{\vec{v} \cdot \vec{w}}{\|v\| \|w\|}\right)$

$$\theta = \cos^{-1}\left(\frac{\vec{r}'(t) \cdot \vec{p}'(t)}{\|\vec{r}'(t)\| \|\vec{p}'(t)\|}\right) = \cos^{-1}\left(\frac{\langle -1, 0, 0 \rangle \cdot \langle 0, 2, 2 \rangle}{(1)(2\sqrt{2})}\right)$$

$$= \cos^{-1}\left(\frac{0 + 0 + 0}{1(2\sqrt{2})}\right) = \cos^{-1}(0) = \pi/2$$