20W-MATH32A-1 Midterm Exam

TOTAL POINTS

43 / 50

QUESTION 1

Problem 1 10 pts

1.1 1a **0 / 2**

- **0 pts** Correct
- ✓ **- 2 pts Incorrect**

1.2 1b **0 / 2**

- **0 pts** Correct
- ✓ **- 2 pts Incorrect**

1.3 1c **2 / 2**

- ✓ **- 0 pts Correct**
- **2 pts** Incorrect

1.4 1d **0 / 2**

- **0 pts** Correct
- ✓ **- 2 pts Incorrect**

1.5 1e **2 / 2**

- ✓ **- 0 pts Correct**
- **2 pts** Incorrect

QUESTION 2

Problem 2 10 pts

2.1 2a **2 / 2**

- ✓ **- 0 pts Correct**
- **2 pts** incorrect

2.2 2b **2 / 2**

- ✓ **- 0 pts Correct**
- **2 pts** Incorrect

2.3 2c **2 / 2**

- ✓ **- 0 pts Correct**
- **2 pts** Incorrect

2.4 2d **2 / 2**

- ✓ **- 0 pts Correct**
- **2 pts** Incorrect

2.5 2e **2 / 2**

- ✓ **- 0 pts Correct**
- **2 pts** Incorrect

QUESTION 3

Problem 3 10 pts

3.1 3a **2 / 2**

- ✓ **- 0 pts Correct**
- **1 pts** Calculate length of v
- **1 pts** Divide v by length
- **0.5 pts** Calculation Error

3.2 3b **3 / 3**

- ✓ **- 0 pts Correct**
- **1 pts** Use Definition of Projection
- **1 pts** Computed projection of v onto w rather than w onto v
- **1 pts** Compute v * w
- **1 pts** Compute v * v (or use |v|^2 from part a)
- **0.5 pts** Calculation Error
- **1 pts** Definition of Dot Product

3.3 3c **5 / 5**

- ✓ **- 0 pts Correct**
- **2 pts** Use a cross product to find a vector normal to the plane
- **1 pts** Identify a point on the plane
- **2 pts** Find equation of plane using a point and a normal vector

- **0.5 pts** Calculation Error
- **1 pts** Definition of Cross Product

QUESTION 4

Problem 4 10 pts

4.1 4a **2 / 2**

✓ **- 0 pts** Correct

- 1 pts incorrect second derivative

- 1 pts incorrect first derivative

- 1.5 pts Incorrect differentiation technique

4.2 4b **3 / 3**

✓ **- 0 pts** Correct

- 1 pts wrong integrand

- 1 pts incorrect integration

- 3 pts Inappropriate technique

- 2 pts Not using arc-length integral formula

4.3 4c **2 / 2**

✓ **- 0 pts** Correct

- 1 pts finding t(s), but not r(t(s))

- 2 pts Incorrect technique

- 1 pts using wrong arc-length formula

- 1 pts Wrong final answer

- 2 pts Blank

4.4 4d **2 / 3**

✓ **+ 1 pts** **Correct curvature formula**

✓ **+ 1 pts** **correct cross product**

+ 1 pts correct vector calculations

+ 1 pts correct conclusion

+ 0 pts Blank

+ 0 pts incorrect approach

+ 0.5 pts incorrect second derivative

QUESTION 5

Problem 5 10 pts

5.1 5a **5 / 5**

✓ **- 0 pts** Correct

- 1 pts Minor error

- 5 pts Click here to replace this description.

5.2 5b **5 / 5**

✓ **- 0 pts** Correct

- 5 pts Wrong approach

- 2 pts wrong sum of cross products

- 1 pts Minor error

Math 32A | Midterm Exam | Winter 2020

Instructor: David Wihr Taylor

Exam date: 10 February 2020

Last name: ____________________ Discussion Section: __

First name: ____________________ Student ID number: __

Instructions

- You have 50 minutes for this exam.
- You must show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Circle or otherwise indicate your final answers. If you use the back of a page, please indicate that you have work on the reverse side.
- You may not use books, mobile phones, or any outside help during this exam. You may not collaborate with anyone in any way during this exam.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

Problem 1

(10 points total) For each of the following parts, circle **T** if the statement is true, or **F** if the statement is false. There is no partial credit, but no penalty for a wrong answer either.

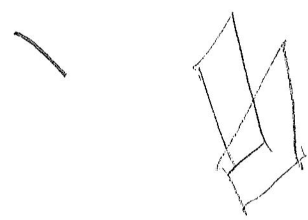
(a) (2 points) For the standard unit vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$, $\left(\hat{\mathbf{i}} \times \left(\hat{\mathbf{i}} \times \left(\hat{\mathbf{i}} \times \hat{\mathbf{j}}\right)\right)\right)$ is parallel to $\left(\hat{\mathbf{i}} \times \hat{\mathbf{j}}\right)$.

$i \perp j$

perpendicular ↑

T F

(b) (2 points) Suppose two planes, $\mathcal{P}_1$ and $\mathcal{P}_2$, are not parallel, with respective normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$. Then, $\mathbf{n}_1 \times \mathbf{n}_2$ is parallel to the intersection of $\mathcal{P}_1$ and $\mathcal{P}_2$.



T F

(c) (2 points) Every vector contained in the line $\mathbf{r}(t) = \langle 1+2t, 1+3t, 1+4t \rangle$ is parallel to the vector $\langle 1, 1, 1 \rangle$.

$\vec{r}(t) = \langle 1,1,1 \rangle + t\langle 2,3,4 \rangle$

T F

(d) (2 points) If the curvature of $\mathbf{r}(t)$ is constant at all times, then the graph of $\mathbf{r}(t)$ is a circle.

T F

(e) (2 points) For vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{R}^3$, $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) \leq \|\mathbf{u}\| \, \|\mathbf{v}\| \, \|\mathbf{w}\|$

$\langle 1,2,3 \rangle \cdot \begin{vmatrix} i & j & k \\ 2 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix}$ $\quad \langle 1,2,3 \rangle \cdot \langle -3,0,3 \rangle$

$-3+9=6$

$\|u\| = \sqrt{1+4+9} = \sqrt{14}$

$-3\hat{i}+3\hat{k}$

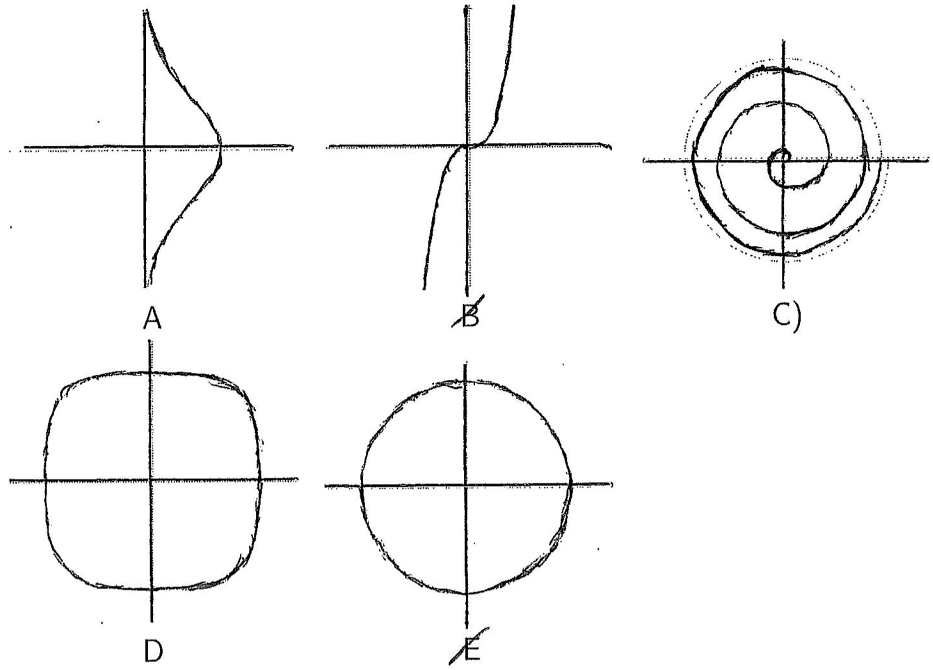
$\|v\| = \sqrt{9} = 3$

$\|w\| = \sqrt{1+4+1} = \sqrt{6}$

T F

Problem 2

(10 points total) Match each vector-valued function with its graph by writing the letter corresponding to the graph in the box adjacent to its formula. There is no partial credit, but no penalty for an incorrect answer either.



(a) (2 points) **E** $\langle \cos(t), \sin(t) \rangle$

(b) (2 points) **A** $\left\langle e^{-t^2}, t \right\rangle$

(c) (2 points) **D** $\left\langle (\cos(t))^{3/5}, (\sin(t))^{3/5} \right\rangle$

(d) (2 points) **B** $\langle t, t^3 \rangle$

(e) (2 points) **C** $\left\langle \frac{t^2}{1+t^2} \cos(8t), \frac{t^2}{1+t^2} \sin(8t) \right\rangle$

Problem 3

This question has parts (a), (b), and (c). Let $\mathbf{v} = 3\hat{\imath} + 4\hat{\jmath} + 1\hat{k}$ and $\mathbf{w} = \hat{\imath} - \hat{\jmath}$.

(a) (2 points) Find the unit vector in the same direction as $\mathbf{v}$.

$$\vec{v} = \langle 3, 4, 1 \rangle$$

$$\hat{e}_v = \frac{\langle 3,4,1 \rangle}{\|\vec{v}\|} = \frac{\langle 3,4,1 \rangle}{\sqrt{9+16+1}} = \boxed{\left\langle \frac{3}{\sqrt{26}}, \frac{4}{\sqrt{26}}, \frac{1}{\sqrt{26}} \right\rangle}$$

(b) (3 points) Compute the projection of $\mathbf{w}$ onto $\mathbf{v}$.

$$\text{proj}_{\vec{v}}\vec{w} = \frac{\vec{v}\cdot\vec{w}}{\vec{v}\cdot\vec{v}}\vec{v} = \frac{\langle 3,4,1 \rangle \cdot \langle 1,-1,0 \rangle}{\langle 3,4,1 \rangle \cdot \langle 3,4,1 \rangle} \langle 3,4,1 \rangle$$

$$= \frac{-1}{9+16+1} \langle 3,4,1 \rangle = \frac{-1}{26} \langle 3,4,1 \rangle$$

$$\text{proj}_{\vec{v}}\vec{w} = \boxed{\left\langle \frac{-3}{26}, \frac{-2}{13}, \frac{-1}{26} \right\rangle}$$

(c) (5 points) Find an equation for the plane which contains $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{0}$ (the zero-vector).

$\langle 0, 0, 0 \rangle$

$$\vec{v} \times \vec{w} = \begin{vmatrix} i & j & k \\ 3 & 4 & 1 \\ 1 & -1 & 0 \end{vmatrix} = \hat{\imath} + \hat{\jmath} - 7\hat{k} = \vec{n} = \langle 1, 1, -7 \rangle$$

$$\boxed{x + y - 7z = 0}$$

Problem 4

(10 points total) Let $\mathbf{r}(t) = \left\langle \frac{3}{5}t, \frac{4}{5}t, \frac{2}{3}t^{\frac{3}{2}} \right\rangle$

(a) (2 points) Find $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$.

$\vec{r}'(t) = \langle \frac{3}{5}, \frac{4}{5}, t^{\frac{1}{2}} \rangle$

$\vec{r}''(t) = \langle 0, 0, \frac{1}{2}t^{-1/2} \rangle$

(b) (3 points) Find the arc-length of $\mathbf{r}(t)$, for $0 \leq t \leq T$. $\frac{2}{3}u^{\frac{1}{2}}$

$u = 1+t$

$du = dt$

$s = g(t) = \int_0^T \|\vec{r}'(t)\| dt = \int_0^T \sqrt{1+t}\, dt = \int \sqrt{u}\, du$

$\|\vec{r}'(t)\| = \sqrt{\frac{9}{25} + \frac{16}{25} + t} = \sqrt{1+t}$

$= \frac{2}{3}u^{\frac{3}{2}}$

$= \frac{2}{3}(1+t)^{\frac{3}{2}} \Big|_0^T$

$= \frac{2}{3}(1+T)^{\frac{3}{2}} - \frac{2}{3}$

~~$= \frac{2}{3}[(1+T)^{3/2} - 1]$~~

$s = 2t$

$g^{-1}(s) = \frac{s}{2}$

(c) (2 points) Find the arc-length parameterization, $\mathbf{r}_1(s)$.

$s = g(t) = \frac{2}{3}(1+t)^{3/2} - \frac{2}{3} = \frac{2}{3}[(1+t)^{3/2} - 1]$

$\frac{3}{2}s = (1+t)^{3/2} - 1$

$\frac{3}{2}s + 1 = (1+t)^{3/2}$

$\left(\frac{3}{2}s+1\right)^{\frac{2}{3}} = 1+t$

$t = \left(\frac{3}{2}s+1\right)^{\frac{2}{3}} - 1$

$g^{-1}(s) = t = \left(\frac{3}{2}s+1\right)^{\frac{2}{3}} - 1$

$\vec{r}(g^{-1}(s)) = \left\langle \frac{3}{5}\left[\left(\frac{3}{2}s+1\right)^{\frac{2}{3}}-1\right], \frac{4}{5}\left[\left(\frac{3}{2}s+1\right)^{\frac{2}{3}}-1\right], \frac{2}{3}\left[\left(\frac{3}{2}s+1\right)^{\frac{2}{3}}-1\right]^{\frac{3}{2}} \right\rangle$

(d) (3 points) Find the curvature of the graph of $\mathbf{r}(t)$ at time $t = 0$.

$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \frac{\langle 3/5, 4/5, t^{1/2} \rangle}{\sqrt{1+t}} = \left\langle \frac{3}{5\sqrt{1+t}}, \frac{4}{5\sqrt{1+t}}, \frac{\sqrt{t}}{\sqrt{1+t}} \right\rangle$

$= \left\langle \frac{3}{5}(1+t)^{-\frac{1}{2}}, \frac{4}{5}(1+t)^{-1/2}, t^{1/2} \times (1+t)^{-1/2} \right\rangle$

$\vec{T}'(t) = \left\langle \frac{-3}{10}(1+t)^{-3/2}, -\frac{4}{10}(1+t)^{-3/2}, \frac{1}{2}t^{-1/2}(1+t)^{-1/2} - \frac{1}{2}t^{1/2}(1+t)^{-3/2} \right\rangle$

$f = t^{1/2}$

$f' = \frac{1}{2}t^{-1/2}$

$g = (1+t)^{-1/2}$

$g' = \frac{1}{2}(1+t)^{-3/2}$

$\vec{T}'(0) = \left\langle \frac{-3}{10}, -\frac{4}{10}, 0 \right\rangle$

$\vec{r}'(t) \times \vec{r}''(t) = \begin{vmatrix} i & j & k \\ 3/5 & 4/5 & t^{1/2} \\ 0 & 0 & \frac{1}{2}t^{-1/2} \end{vmatrix}$

↑ solve

$= \frac{2}{5}t^{-1/2}i - \frac{3}{10}t^{-1/2}j + 0k$ ← OK

$\kappa(t) = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3} = \frac{\frac{1}{2}\sqrt{t}}{(\sqrt{1+t})^3}$

$\|\vec{r}'(t) \times \vec{r}''(t)\| = \sqrt{\frac{4}{25t} + \frac{9}{100t}} = \sqrt{\frac{25}{100t}}$

$= \frac{1}{2}\sqrt{t}$

$\kappa(0) = 0$

Problem 5

(10 points total) This problem has parts (a) and (b). Consider the following four points in $\mathbb{R}^3$: $A = (0,0,0), B = (0,1,1), C = (1,0,0)$, and $D = (2,1,1)$

(a) (5 points) Using cross-products and dot-products, show that these four points lie in a single plane. [Hint: it would help to find the plane that contains any three of the points.]

$\vec{AB} = \langle 0, 1, 1 \rangle$

$\vec{AC} = \langle 1, 0, 0 \rangle$

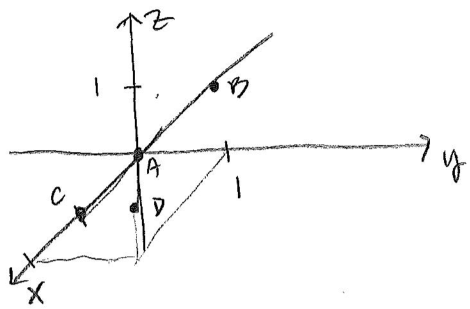
$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \hat{j} - \hat{k} = \langle 0, 1, -1 \rangle = \vec{n}$

$y - z = 0$ contains points A, B, and C

Plug-in Point D : $1 - 1 = 0$ ✓

∴ Points A, B, C, and D all lie on a single plane. This plane could have the equation $y - z = 0$.

(b) (5 points) Using methods we have learned in this class, find the area of the quadrilateral $ABCD$ that these four points define within the plane that contains them.



$\vec{AC} = \langle 1,0,0 \rangle$

$\vec{AD} = \langle 2,1,1 \rangle$

$\vec{AB} = \langle 0,1,1 \rangle$

$$\frac{\|\vec{AC} \times \vec{AD}\|}{2} + \frac{\|\vec{AB} \times \vec{AD}\|}{2} = \frac{\sqrt{2}}{2} + \frac{2\sqrt{2}}{2} = \boxed{\frac{\sqrt{2}}{2} + \sqrt{2}}$$

$$\vec{AC} \times \vec{AD} = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 2 & 1 & 1 \end{vmatrix} = -\hat{j} + k = \langle 0,-1,1 \rangle$$

$$\|\vec{AC} \times \vec{AD}\| = \sqrt{2}$$

$$\vec{AB} \times \vec{AD} = \begin{vmatrix} i & j & k \\ 0 & 1 & 1 \\ 2 & 1 & 1 \end{vmatrix} = 2\hat{j} - 2\hat{k} = \langle 0,2,-2 \rangle$$

$$\|\vec{AB} \times \vec{AD}\| = \sqrt{4+4} = 2\sqrt{2}$$

Formula Sheet

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} a_1 & b_3 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$\mathbf{u}_{\parallel} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|}\right)\frac{\mathbf{v}}{\|\mathbf{v}\|}$$
$$= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|}\right)\mathbf{e}_{\mathbf{v}}$$

$$\mathbf{u} = \mathbf{u}_{\perp\mathbf{v}} + \mathbf{u}_{\parallel\mathbf{v}}$$

$$\|\mathbf{v}\cdot\mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\|\cos\theta$$

$$\|\mathbf{v}\times\mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\|\sin\theta$$

$$\|\mathbf{v}\times\mathbf{w}\|^2 = \|\mathbf{v}\|^2\|\mathbf{w}\|^2 - (\mathbf{v}\cdot\mathbf{w})^2$$

$$\mathbf{n}\cdot\langle x, y, z\rangle = d$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$ax + by + cz = d$$

$$\frac{d}{dt}\mathbf{r}(g(t)) = g'(t)\mathbf{r}'(g(t))$$

$$\frac{d}{dt}(f(t)\mathbf{r}(t)) = f'(t)\mathbf{r}(t) + f(t)\mathbf{r}'(t)$$

$$\frac{d}{dt}(\mathbf{r}_1(t)\cdot\mathbf{r}_2(t)) = \mathbf{r}_1'(t)\cdot\mathbf{r}_2(t) + \mathbf{r}_1(t)\cdot\mathbf{r}_2'(t)$$

$$\frac{d}{dt}(\mathbf{r}_1(t)\times\mathbf{r}_2(t)) = \mathbf{r}_1'(t)\times\mathbf{r}_2(t) + \mathbf{r}_1(t)\times\mathbf{r}_2'(t)$$

$$\mathbf{L}(t) = \mathbf{r}(t_0) + t\mathbf{r}'(t_0)$$

$$\int \mathbf{r}(t)dt = \mathbf{R}(t) + \mathbf{c}$$

$$\int_a^b \mathbf{r}(t)dt = \mathbf{R}(b) - \mathbf{R}(a)$$

$$s(t) = \int_a^t \|\mathbf{r}'(u)\|du$$

$$v(t) = \frac{ds}{dt} = \|\mathbf{r}'(t)\|$$

$$\mathbf{v}(t) = \mathbf{r}'(t)$$

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|}$$

$$\kappa(s) = \left\|\frac{d\mathbf{T}}{ds}\right\|$$

$$\kappa(t) = \frac{1}{v(t)}\left\|\frac{d\mathbf{T}}{dt}\right\|$$

$$\kappa(t) = \frac{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$$

$$\mathbf{B}(t) = \mathbf{T}(t)\times\mathbf{N}(t)$$

$$a_{\mathrm{T}} = v'(t) = \mathbf{a}\cdot\mathbf{T} = \frac{\mathbf{a}\cdot\mathbf{v}}{\|\mathbf{v}\|}$$

$$a_{\mathrm{T}}\mathbf{T} = \left(\frac{\mathbf{a}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}$$

$$a_{\mathrm{N}} = \kappa(t)v(t)^2 = \sqrt{\|\mathbf{a}\|^2 - |a_{\mathrm{T}}|^2}$$

$$a_{\mathrm{N}}\mathbf{N} = \mathbf{a} - a_{\mathrm{T}}\mathbf{T} = \mathbf{a} - \left(\frac{\mathbf{a}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}$$