MATH 32A: Calculus of Several Variables
Lecture 4, Fall 2018
Mid-term exam 1

University of California, Los Angeles

Instructor: David Arnold

Total time: 40 minutes

Question	Marks	Score
1	4	4
2	4	3
3	7	7
4	5	5
Total:	20	19

First Name:

Last Name:

ID number:

Signature:

Date: 10-29-18

By signing this form you declare that you will act in accordance with the UCLA Student Conduct Code.

Please circle your discussion section

	Zach	Patrick	Bertrand
Tuesday	4A	4C	4E
Thursday	4B	4D	4F

Instructions

- Notes/textbooks may not be used
- Simple scientific calculators only may be used
- Mobile phones must be switched off, and placed in closed bags on the floor
- There is a blank page at the end of the exam booklet for extra space for answers
- Answers are graded based on correctness and explanations/calculations

1. (a) Indicate whether the statements are True or False, and give a brief justification (a few words is sufficient but you must write something).

i. (1 mark) $\mathbf{r}(t) = (1, 0, t^3)$ is an arc length paramaterisation.

True False

$r'(t) = (0,0,3t^2)$
$|r'(t)| = \sqrt{0^2+0^2+(3t^2)^2}$
$= \sqrt{9t^4}$
$= 3t^2$

Arc length parameterizations have $|r'(t)| = 1$, and $3t^2 \neq 1$.

ii. (1 mark) $|\mathbf{v} \cdot (\mathbf{u} \times \mathbf{w})| = |\mathbf{w} \cdot (\mathbf{v} \times \mathbf{u})|$ for all nonzero vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$.

True False

$|\vec{v} \cdot (\vec{u} \times \vec{w})|$ is the volume for a parallelepiped, and yields the same result regardless of the order of the vectors since volume is always positive.

(b) (2 marks) Find the constant λ such that $\lambda\mathbf{v} + \mathbf{w}$ is perpendicular to $\mathbf{u}$, where $\mathbf{v} = (2, -1)$, $\mathbf{w} = (1, 3)$, and $\mathbf{u} = (10, 2)$.

The dot product of two perpendicular vectors is 0, so

$(\lambda\vec{v} + \vec{w}) \cdot u = 0$

$\lambda\vec{v} \cdot \vec{u} + \vec{w} \cdot \vec{u} = 0$

$\lambda(\vec{v} \cdot \vec{u}) + \vec{w} \cdot \vec{u} = 0$

$\lambda[(2,-1) \cdot (10,2)] + (1,3) \cdot (10,2) = 0$

$\lambda(2(10) + (-1)(2)) + (1)(10) + (3)(2) = 0$

$\lambda 18 + 16 = 0$

$\lambda = \frac{-16}{18}$

$\boxed{= -\frac{8}{9}}$

2. (4 marks) Simplify the following

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{f}(t)\cdot(\mathbf{g}(t)\times\mathbf{h}(t))\right),$$

assuming $\mathbf{f}(t)$ is parallel to $\mathbf{h}'(t)$ and $\mathbf{g}'(t)$ is parallel to $\mathbf{h}(t)$.

$$\frac{d}{dt}(f(t)\cdot(g(t)\times h(t)))$$

$$= \frac{d}{dt}(f(t))\cdot(g(t)\times h(t)) + f(t)\cdot\frac{d}{dt}(g(t)\times h(t))$$

$$= f'(t)\cdot(g(t)\times h(t)) + f(t)\cdot\left(g(t)\times\frac{d}{dt}(h(t)) + \frac{d}{dt}(g(t))\times h(t)\right)$$

$$= f'(t)\cdot(g(t)\times h(t)) + f(t)\cdot(g(t)\times h'(t) + \underbrace{\cancel{g'(t)\times h(t)}}_{}{}^{\to 0})$$

Since $g'(t) \parallel h(t)$, $g'(t)\times h(t) = 0$

$$= f'(t)\cdot(g(t)\times h(t)) + \underbrace{\cancel{f(t)\cdot(g(t)\times h'(t))}}_{\text{since } f}{}^{\to 0}$$

why?

-1

$\cancel{f'(t)}\cdot(g(t)\times h(t))$

$$\cancel{= \begin{vmatrix} f_x'(t) & f_y'(t) & f_z'(t) \\ g_x(t) & g_y(t) & g_z(t) \\ h_x(t) & h_y(t) & h_z(t) \end{vmatrix} = f_x'(t)(g_y(t)h_z(t) - g_z(t)h_y(t)) - f_y'(t)(g_x(t)h_z(t) - g_z(t)h_x(t)) + f_z'(t)(g_x(t)h_y(t)}$$

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3. Consider the points $P = (2, -3, 2)$, $Q = (3, -2, 2)$, $R = (2, -2, 4)$, and $S = (4, 1, 6)$.

(a) (4 marks) Find the equation of the plane containing P, Q, and S.

$\vec{PQ} = (3-2, -2-{-3}, 2-2)$
$= (1, 1, 0)$
$\vec{PS} = (4-2, 1-{-3}, 6-2)$
$= (2, 4, 4)$

Normal vector to the plane is $\vec{PQ} \times \vec{PS}$:

$\begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 2 & 4 & 4 \end{vmatrix} = i(4-0) - j(4-0) + k(4-2)$
$= (4, -4, 2)$

$4x - 4y + 2z = d$
$d = 4(2) - 4(-3) + 2(2)$
$= 24$
$\rightarrow 4x - 4y + 2z = 24$

(b) (2 marks) Find the angle between $\overrightarrow{PQ}$ and $\overrightarrow{PS}$.

$\|\vec{PQ} \times \vec{PS}\| = \|\vec{PQ}\|\|\vec{PS}\| \sin\theta$

$\sin\theta = \dfrac{\|\vec{PQ} \times \vec{PS}\|}{\|\vec{PQ}\|\|\vec{PS}\|}$

$\|\vec{PQ}\| = \sqrt{1^2+1^2+0^2} = \sqrt{2}$
$\|\vec{PS}\| = \sqrt{2^2+4^2+4^2} = 6$
$\|\vec{PQ} \times \vec{PS}\| = \sqrt{4^2+(-4)^2+2^2}$
$= 6$

$\sin\theta = \dfrac{6}{6(\sqrt{2})}$

$\sin\theta = \dfrac{1}{\sqrt{2}}$

$\theta = 45°$

(c) (1 mark) Find the equation of the plane containing R that is parallel to the plane you found in part (a).

Parallel planes have the same normal vector
$4x - 4y + 2z = d$
$d = 4(2) - 4(-2) + 2(4)$
$= 24$
$\rightarrow 4x - 4y + 2z = 24$

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4. (a) (2 marks) Find the speed of particle whose position at time t is $\mathbf{r}(t) = (1, 2, 3) + t\,(2, 3, 6)$.

$r(t) = (1, 2, 3) + t(2, 3, 6)$

$= (1+2t,\ 2+3t,\ 3+6t)$

$r'(t) = (2, 3, 6)$

$|r'(t)| = \sqrt{2^2+3^2+6^2} = \boxed{7}$

(b) (3 marks) Consider a parameterisation $\mathbf{r}(t)$ with constant speed v (that is, with $v = ||\mathbf{r}'(t)||$ constant). By finding the arc-length equation, determine an arc-length parameterisation of $\mathbf{r}(t)$.

$||r'(t)|| = 7$

$s = \int_0^t ||r'(u)||\, du$

$s = \int_0^t 7\, du$

$s = 7t$

$t = g^{-1}(s) = \frac{s}{7}$

$r(s) = \left(1+2\left(\frac{s}{7}\right), 2+3\left(\frac{s}{7}\right), 3+6\left(\frac{s}{7}\right)\right)$

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End of exam