1. (a) Indicate whether the statements are True or False, and give a brief justification (a few words is sufficient but you must write something).

i. (1 mark) $\mathbf{r}(t) = (1, 0, t^3)$ is an arc length paramaterisation.

True   False

Arc length paramat if $\| r_1'(t) \| = 1$

$\| r_1'(t) \| = \| (0, 0, 3t^2) \| = \underline{3t^2} \neq 1$ $\nearrow$ so $r(t)$ is not ALP

ii. (1 mark) $|\mathbf{v} \cdot (\mathbf{u} \times \mathbf{w})| = |\mathbf{w} \cdot (\mathbf{v} \times \mathbf{u})|$ for all nonzero vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$.

True   False

Given that $|v \cdot (u \times w)|$ & $|w \cdot (v \times u)|$ are absolute values & both give volume of parallel-eliped, they are equal.

(b) (2 marks) Find the constant λ such that $\lambda\mathbf{v} + \mathbf{w}$ is perpendicular to $\mathbf{u}$, where $\mathbf{v} = (2, -1)$, $\mathbf{w} = (1, 3)$, and $\mathbf{u} = (10, 2)$.

$(\lambda \underset{\sim}{v} + \underset{\sim}{w}) \cdot (\underset{\sim}{u}) = 0$ $\rightarrow$ dot product is 0 when two vectors perp. ($\perp$)

$(\lambda(2,-1) + (1,3)) \cdot (10,2) = 0$

$((2\lambda, -\lambda) + (1,3)) \cdot (10,2) = 0$

$(2\lambda + 1, 3 - \lambda) \cdot (10,2) = 0$

$10(2\lambda + 1) + 2(3 - \lambda) = 0$

$20\lambda + 10 + 6 - 2\lambda = 0$

$18\lambda = -16$

$\lambda = -\frac{16}{18}$

$\nearrow$ $\boxed{\lambda = -\frac{8}{9}}$

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2. (4 marks) Simplify the following

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{f}(t)\cdot(\mathbf{g}(t)\times\mathbf{h}(t))\right).$$

assuming $\mathbf{f}(t)$ is parallel to $\mathbf{h}'(t)$ and $\mathbf{g}'(t)$ is parallel to $\mathbf{h}(t)$.

$$\frac{d}{dt}\left(f(t)\cdot(g(t)\times h(t))\right)$$

$$= f'(t)\cdot(g(t)\times h(t)) + f(t)\cdot(g(t)\times h'(t) + \overset{=0}{g'(t)\times h(t)})$$

$f(t) \parallel h'(t)$, so $f(t)\times h'(t) = 0$

$g'(t) \parallel h(t)$, so $g'(t)\times h(t) = 0$

$$= f'(t)\cdot(g(t)\times h(t)) + f(t)\cdot(g(t)\times h'(t))$$

$$= f'(t)\cdot(g(t)\times h(t)) + g(t)\cdot(\underbrace{f(t)\times h'(t)}_{=0})$$

$g(t)\cdot 0 = 0.$

$$= \boxed{f'(t)\cdot(g(t)\times h(t))}$$

$f(t)$ & $g(t)$ can be switched

They can, but the sign changes.

$f\cdot(g\times h') = -g\cdot(f\times h')$ $=0$

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3. Consider the points $P = (2, -3, 2)$, $Q = (3, -2, 2)$, $R = (2, -2, 4)$, and $S = (4, 1, 6)$.

(a) (4 marks) Find the equation of the plane containing P, Q, and S.

Verified work on extra paper

$\underset{\sim}{PQ} = (3,-2,2) - (2,-3,2) = (1,1,0)$

$\underset{\sim}{SQ} = (4,1,6) - (3,-2,2) = (1,3,4)$

Find normal vect. of $\underset{\sim}{PQ}$ & $\underset{\sim}{SQ}$ to find norm. v of plane

$$\text{dot}\begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 1 & 3 & 4 \end{vmatrix} = (4-0)i - (4-0)j + (3-1)k$$

$$= 4i - 4j + 2k \qquad (4,-4,2)$$

Coeff for plane eq

$4x - 4y + 2z = d$ ← plug in S

$\underset{16}{4(4)} - \underset{-4}{4(1)} + \underset{12}{2(6)} = d$, so $\boxed{4x - 4y + 2z = 24}$

(b) (2 marks) Find the angle between $\overrightarrow{PQ}$ and $\overrightarrow{PS}$.

$$\cos\theta = \frac{\underset{\sim}{PQ} \cdot \underset{\sim}{PS}}{\|\underset{\sim}{PQ}\|\,\|\underset{\sim}{PS}\|} = \frac{(1,1,0)\cdot(2,4,4)}{(\sqrt{1+1+0})(\sqrt{4+16+16})}$$

$$= \frac{2+4}{\sqrt{2}\sqrt{36}} = \frac{6}{6\sqrt{2}}$$

$\theta = \arccos\left(\frac{1}{\sqrt{2}}\right)$

$\boxed{\theta = \frac{\pi}{4}}$

$\underset{\sim}{PQ} = (1,1,0)$

$\underset{\sim}{PS} = (4,1,6) - (2,-3,2) = (2,4,4)$

$\sqrt{2}$ 1 1

(c) (1 mark) Find the equation of the plane containing R that is parallel to the plane you found in part (a).

If parallel, normal vector is same, so $4x - 4y + 2z = d$. Plug in R to find d & plane eq.

$\underset{8}{4(2)} - \underset{+8}{4(-2)} + \underset{+8}{2(4)} = d \qquad d = 24$

$\boxed{4x - 4y + 2z = 24}$

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4. (a) (2 marks) Find the speed of particle whose position at time t is $\mathbf{r}(t) = (1,2,3) + t\,(2,3,6)$.

$$\text{speed} = \|r'(t)\| \qquad r(t) = (2t+1, 3t+2, 6t+3)$$
$$= \|(2,3,6)\|$$
$$= \sqrt{4+9+36}$$
$$= \sqrt{13+36} = \sqrt{49} = \boxed{7} \checkmark$$

(b) (3 marks) Consider a parameterisation $\mathbf{r}(t)$ with constant speed v (that is, with $v = ||\mathbf{r}'(t)||$ constant). By finding the arc-length equation, determine an arc-length parameterisation of $\mathbf{r}(t)$.

$$g(t) = \int_0^t \|r'(t)\|\,dt$$
$$= \int_0^t 7\,dt = 7t \checkmark$$
$$g^{-1}(t) = \frac{t}{7}, \checkmark$$

so arc-length parameter. is

$$\left(1+2\left(\tfrac{t}{7}\right), 2+3\left(\tfrac{t}{7}\right), 3+6\left(\tfrac{t}{7}\right)\right)$$

Check:

$$\left\|\left(\tfrac{2}{7}, \tfrac{3}{7}, \tfrac{6}{7}\right)\right\|$$
$$= \sqrt{\frac{4}{49}+\frac{9}{49}+\frac{36}{49}}$$
$$= \sqrt{\frac{49}{49}} = 1$$

Since speed is 1, the arc-length param. is correct

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