Math 32A - Winter 2019

Practice Exam 2

Full Name: Solutions

UID: ______________________

Circle the name of your TA and the day of your discussion:

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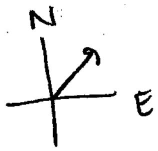
Tuesday   Thursday

Instructions:

- Read each problem carefully.
- Show all work clearly and circle or box your final answer where appropriate.
- Justify your answers. A correct final answer without valid reasoning will not receive credit.
- Simplify your answers as much as possible.
- Include units with your answer where applicable.
- Calculators are not allowed but you may have a 3×5 inch notecard.

Page	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

1. Suppose the elevation of a mountain above sea level at a point (x, y) is given by the function $z = 2000 - 2x^2 - 4y^2$ feet, where the positive x-axis points east, and the positive y-axis points north. A climber is at the point $P = (-20, 5, 1100)$.



(a) (10 points) Suppose the climber uses a compass to walk northeast from P. At what rate will the climber initially ascend or descend?

$\vec{u} = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle \quad D_{\vec{u}} f(-20, 5) = ? \qquad D_{\vec{u}} f = \nabla f \cdot \vec{u}$

$\nabla f = \langle f_x, f_y \rangle = \langle -4x, -8y \rangle$

$\nabla f(-20, 5) = \langle 80, -40 \rangle$

$D_{\vec{u}} f = \nabla f \cdot \vec{u} = \langle 80, -40 \rangle \cdot \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle = \frac{80\sqrt{2}}{2} - \frac{40\sqrt{2}}{2} = \frac{40\sqrt{2}}{2} = 20\sqrt{2}$

this is positive so the climber will ascend at a rate of $20\sqrt{2}$ feet up per foot traveled horizontally.

(b) (5 points) Suppose the climber wants to travel down the mountain from the point P as quickly as possible. In what direction should the climber set out from P? Find a unit vector in this direction.

$\nabla f = \langle 80, -40 \rangle$ ← direction of steepest increase

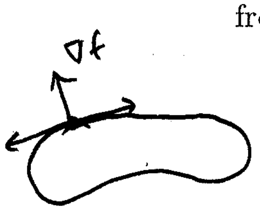
$-\nabla f = \langle -80, 40 \rangle$ ← direction of steepest decrease

$= 40 \langle -2, 1 \rangle$ so in direction of $\langle -2, 1 \rangle$

$\| \langle -2, 1 \rangle \| = \sqrt{4+1} = \sqrt{5}$

$\boxed{\vec{u} = \langle \frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \rangle}$

(c) (5 points) Find all possible directions the climber could walk to travel a level path from P (neither ascending nor descending). Give your answers as unit vectors.



Should travel tangent to a level curve, i.e. orthogonal to ∇f or orthogonal to $\langle 2, -1 \rangle$

So along $\langle a, b \rangle$ with $\langle a, b \rangle \cdot \langle 2, -1 \rangle = 0$

$2a - b = 0$

$2a = b$

if a unit vector $a^2 + b^2 = 1$

$a^2 + (2a)^2 = 1$

$5a^2 = 1$

$a^2 = \frac{1}{5} \qquad a = \pm \frac{1}{\sqrt{5}}$

$\boxed{\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle \text{ and } \langle \frac{-1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \rangle}$

/20

2. (10 points) Show that every plane that is tangent to the cone $z^2 = x^2+y^2$ passes through the origin.

Find the tangent plane at (a,b,c):

$F(x,y,z) = x^2+y^2-z^2=0$

$\nabla F = \langle F_x, F_y, F_z \rangle = \langle 2x, 2y, -2z \rangle$

$\vec{n} = \nabla F(a,b,c) = \langle 2a, 2b, -2c \rangle$

tangent plane : $2a(x-a)+2b(y-b)-2c(z-c)=0$

If $(x,y,z) = (0,0,0)$

Is $2a(0-a)+2b(0-b)-2c(0-c) \overset{??}{=} 0$?

$-2a^2-2b^2+2c^2 \overset{?}{=} 0$ yes, b/c

$2(a^2+b^2-c^2)=0$ for any (a,b,c) on the cone. ✓

So $(0,0,0)$ is on any tangent plane.

3. (10 points) At a landscaping firm, a dirt pile starts as a cone with radius 10 meters and height 9 meters. As it is used up, the height decreases by 0.3 meters per day, but due to slippage the radius increases by 0.1 meters per day. What was the initial rate of dirt usage by the firm in cubic meters per day? *Hint:* You may use the fact that the volume of a cone is is $V = \frac{1}{3}\pi r^2 h$.

$V = \frac{1}{3}\pi r^2 h$ $\quad r=r(t),\ h=h(t)$ $\quad \frac{dV}{dt} = ?$

$\frac{dr}{dt} = +0.1 \frac{m}{day}$ $\quad \frac{dh}{dt} = -0.3 \frac{m}{day}$ $\quad$ At $t=0$ $r=10m$, $h=9m$

$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt}$

$= \left(\frac{2}{3}\pi r h\right)\frac{dr}{dt} + \left(\frac{1}{3}\pi r^2\right)\frac{dh}{dt}$ $\quad 0.1 = \frac{1}{10},\ 0.3 = \frac{3}{10}$

$= \frac{2}{3}\pi(10)(9)(0.1) + \frac{1}{3}\pi(10)^2(-0.3)$

$= 6\pi - 10\pi$

$= -4\pi\ m^3/day$ $\quad$ So usage is initially $\boxed{4\pi\ m^3/day}$

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4. (10 points) Show the following limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^6}{xy^3}$$

$f(x,y) = \frac{x^2 - y^6}{xy^3}$

Try different paths. Notice we can't evaluate at $x=0$ or $y=0$ b/c of denominator

Along $y=x$: $\lim_{x\to 0} f(x,x) = \lim_{x\to 0} \frac{x^2 - x^6}{x^4} = \lim_{x\to 0} \frac{1-x^4}{x^2} = \infty$

(already DNE but would be ∞)

Along $x=y^3$: $\lim_{y\to 0} f(y^3,y) = \lim_{y\to 0} \frac{y^6 - y^6}{y^6} = \lim_{y\to 0} \frac{0}{y^6} = 0$

$0 \neq \infty$ so limit DNE.

5. (10 points) Show the following limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 + \sin^2 y}{2x^2 + y^2}$$

$f(x,y) = \frac{x^2 + \sin^2 y}{2x^2 + y^2}$

Again try different paths.

Along $y=0$ $\lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{x^2 + \sin^2(0)}{2x^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}$

Along $x=0$ $\lim_{y\to 0} f(0,y) = \lim_{y\to 0} \frac{\sin^2 y}{y^2}$ $\frac{0}{0}$

$\overset{\text{L'Hospital's}}{=} \lim_{y\to 0} \frac{2\sin(y)\cos(y)}{2y}$ $\frac{0}{0}$

$\overset{\text{L'Hospital's}}{=} \lim_{y\to 0} \frac{\cos^2 y - \sin^2 y}{1} = \frac{1-0}{1} = 1$

$1 \neq \frac{1}{2}$ so limit DNE

/20

6. Consider the curve $\mathbf{r}(t) = \langle t^2, \frac{2}{3}t^3, t\rangle$.

(a) (6 points) Find the length of the curve for $1 \le t \le 3$.

$\vec{r}\,'(t) = \langle 2t, 2t^2, 1\rangle$ $\quad \|\vec{r}\,'(t)\| = \sqrt{4t^2+4t^4+1} = \sqrt{(1+2t^2)^2}$

$\|\vec{r}\,'(t)\| = 1+2t^2$ (always positive)

$L = \int_1^3 \|\vec{r}\,'(t)\|\,dt = \int_1^3 1+2t^2\,dt = t + \frac{2}{3}t^3\Big]_1^3$

$= 3 + \frac{2}{3}(27) - 1 - \frac{2}{3} = 2 + 18 - \frac{2}{3} = 20 - \frac{2}{3} = \boxed{\frac{58}{3}}$

(b) (10 points) Find the $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$-frame for $\mathbf{r}(t)$ at the point $(1, \frac{2}{3}, 1)$.

$\vec{r}\,'(t) = \langle 2t, 2t^2, 1\rangle$ $\quad \|\vec{r}\,'(t)\| = 1+2t^2$ $\quad t = 1$

$\vec{T}(t) = \langle \frac{2t}{1+2t^2}, \frac{2t^2}{1+2t^2}, \frac{1}{1+2t^2}\rangle$ at $t=1$ $\boxed{\vec{T}(1) = \langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3}\rangle}$

$\vec{T}\,'(t) = \langle \frac{(1+2t^2)(2) - (2t)(4t)}{(1+2t^2)^2}, \frac{(1+2t^2)(4t) - (2t^2)(4t)}{(1+2t^2)^2}, \frac{-4t}{(1+2t^2)^2}\rangle$

$= \langle \frac{2-4t^2}{(1+2t^2)^2}, \frac{4t}{(1+2t^2)^2}, \frac{-4t}{(1+2t^2)^2}\rangle$

at $t=1$ $\quad \vec{T}\,'(1) = \langle \frac{-2}{9}, \frac{4}{9}, \frac{-4}{9}\rangle$ $\quad \|\vec{T}\,'(1)\| = \sqrt{\frac{4}{81} + \frac{16}{81} + \frac{16}{81}} = \sqrt{\frac{36}{81}} = \frac{6}{9} = \frac{2}{3}$

$\vec{N}(1) = \frac{\vec{T}\,'(1)}{\|\vec{T}\,'(1)\|} = \boxed{\langle -\frac{1}{3}, \frac{2}{3}, \frac{-2}{3}\rangle = \vec{N}(1)}$

$\vec{B} = \vec{T}\times\vec{N} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2/3 & 2/3 & 1/3 \\ -1/3 & 2/3 & -2/3 \end{vmatrix} = \langle \frac{-4}{9} - \frac{2}{9}, -(\frac{-4}{9} + \frac{1}{9}), \frac{4}{9} + \frac{2}{9}\rangle$

$\boxed{\vec{B}(1) = \langle \frac{-2}{3}, \frac{1}{3}, \frac{2}{3}\rangle}$

(c) (4 points) Find the curvature of $\mathbf{r}(t)$ at the point $(1, \frac{2}{3}, 1)$.

$\kappa(1) = \frac{\|\vec{T}\,'(1)\|}{\|\vec{r}\,'(1)\|} = \frac{2/3}{3} = \frac{2}{3}\cdot\frac{1}{3} = \boxed{\frac{2}{9}}$

/20

7. (20 points) Match each function with its graph on the next and its contour plot on the following page.

1. $f(x, y) = \sin(y)$ i, F

2. $f(x, y) = (x^2 - y^2)^2$ ii, D

3. $f(x, y) = (x - y)^2$ vi, G

4. $f(x, y) = 3 - x^2 - y^2$ vii, E

5. $f(x, y) = \sin(4x)e^{-x^2-y^2}$ v, C

6. $f(x, y) = \sin(x)\sin(y)e^{-x^2-y^2}$ iii, A

7. $f(x, y) = \dfrac{x}{1 + x^2 + y^2}$ iv, H

8. $f(x, y) = \dfrac{1}{1 + x^2 + y^2}$ viii, B

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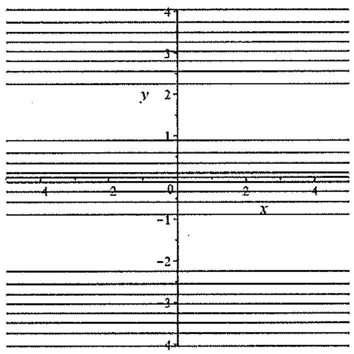
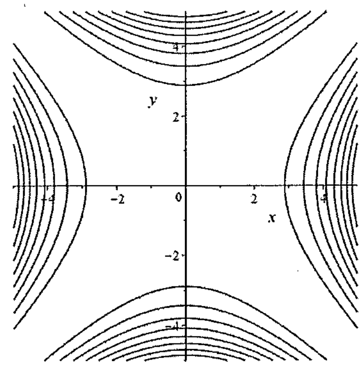
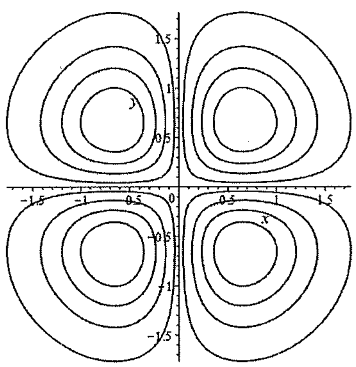


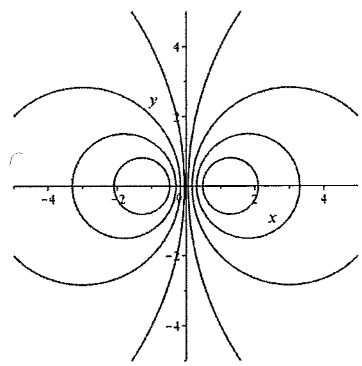
(i)



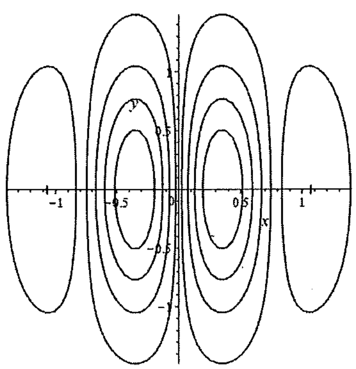
(ii)



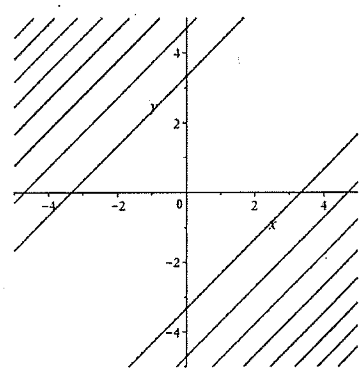
(iii)



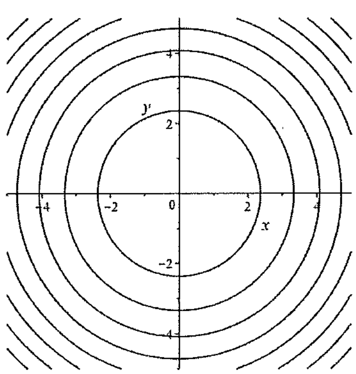
(iv)



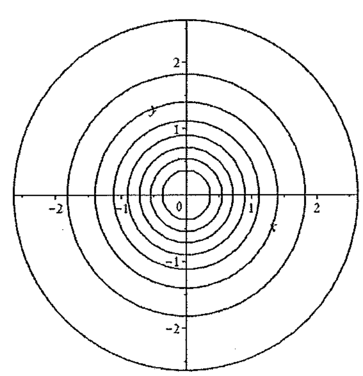
(v)



(vi)



(vii)



(viii)

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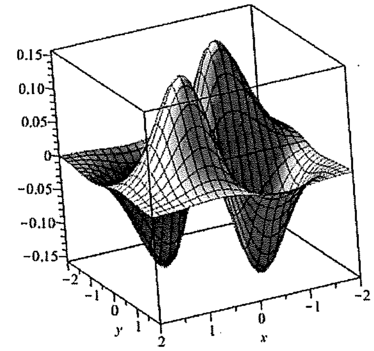
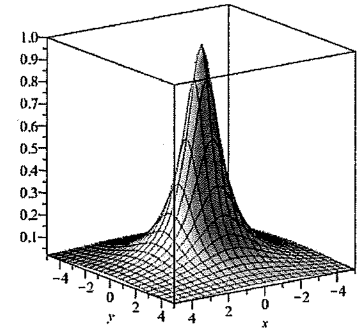
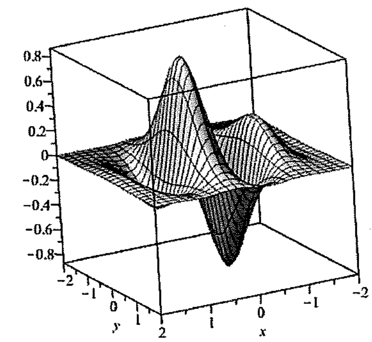

0.15 0.10 0.05 0 -0.05 -0.10 -0.15
-2 -1 0 1 2
y
2 1 0 -1 -2
x
(A)


1.0 0.9 0.8 0.7 0.6 0.5 0.4 0.3 0.2 0.1
-4 -2 0 2 4
y
4 2 0 -2 -4
x
(B)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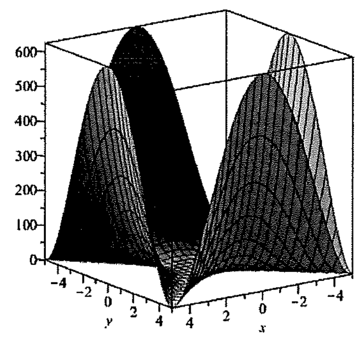
-2 -1 0 1 2
y
2 1 0 -1 -2
x
(C)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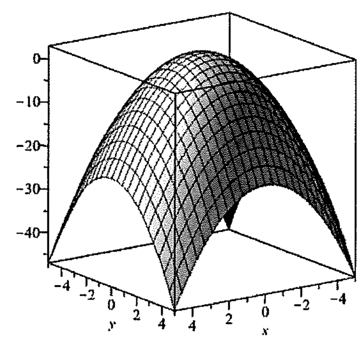
-4 -2 0 2 4
y
4 2 0 -2 -4
x
(D)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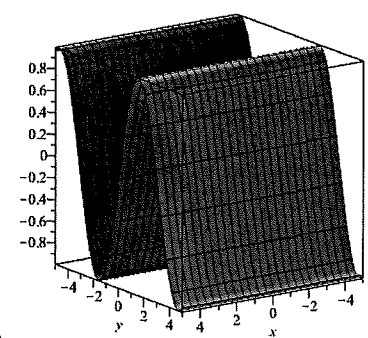
y
4 2 0 -2 -4
x
(E)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y
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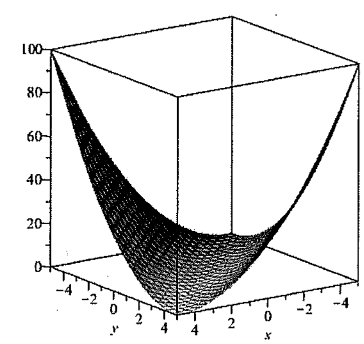
x
(F)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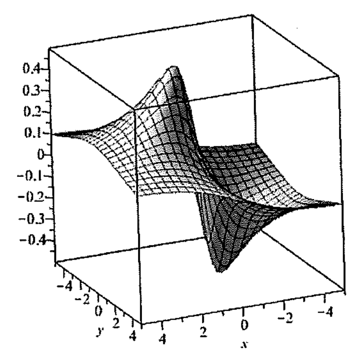
-4 -2 0 2 4
y
4 2 0 -2 -4
x
(G)


0.4 0.3 0.2 0.1 0 -0.1 -0.2 -0.3 -0.4
-4 -2 0 2 4
y
4 2 0 -2 -4
x
(H)

/0