1. (20 points) Consider the surface defined by $ze^{2x} + x^2y + y = 3 + 2e^{z+1}$.

(a) Find an equation of the tangent plane to the surface at the point $P = (0, 5, -1)$.

$ze^{2x} + x^2y + y - 2e^{z+1} - 3 = 0$

$\nabla \vec{F} = \langle F_x, F_y, F_z \rangle$

$\nabla \vec{F} = \langle 4e^{2x} + 2xy, x^2+1, e^{2x} - 2e^{z+1} \rangle$

$\nabla \vec{F}(0,5,-1) = \langle 4, 1, -1 \rangle$

$4(x-0) + 1(y-5) - 1(z+1) = 0$

$4x + y - 5 - z - 1 = 0$

$4x + y - z = 6$

(b) Find a vector equation for the line passing through the surface at $P = (0, 5, -1)$ orthogonal to the plane found in part (a).

If orthogonal to the plane, the direction vector is ∇F because ∇F is normal to the plane.

$\rightarrow \nabla F = \langle 4, 1, -1 \rangle$

$\vec{r}(t) = \langle 0, 5, -1 \rangle + t\langle 4, 1, -1 \rangle$

2. (12 points) Either give an example of a function $f(x, y)$ with $f_x(x, y) = 2x + y^2e^x$ and $f_y(x, y) = x^2 + y^2e^x$ or show that no such function f can exist.

$f_{xy}(x,y) = 2ye^x$

$f_{yx}(x,y) = 2x + y^2e^x$

Since $f_{xy} \neq f_{yx}$, which results in lack of continuity of these partials, no such function f can exist by Clairaut's theorem.

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3. (16 points) Reparameterize the curve $\mathbf{r}(t) = \langle \sqrt{15}\,t^2, \cos(t^2), \sin(t^2) \rangle$ where $t \geq 0$ with respect to arc length.

$\vec{r}\,'(t) = \langle 2\sqrt{15}\,t, -2t\sin(t^2), 2t\cos(t^2) \rangle$

$\|r'(t)\| = \sqrt{(2\sqrt{15}t)^2 + (-2t\sin(t^2))^2 + (2t(\cos t^2))^2} = \sqrt{60t^2+4t^2} = \sqrt{64t^2} = 8t$

$s(t) = \int_0^t \|r'(u)\|\,dt = \int_0^t 8t\,dt = 4t^2$

$s = 4t^2$

$t^2 = \frac{s}{4}$ ← sub into original

$\vec{r}(s) = \langle \frac{\sqrt{15}\,s}{4}, \cos(\frac{s}{4}), \sin(\frac{s}{4}) \rangle$

4. (10 points) Show the following limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+3y^4}$$

set $x = y^2$

$\lim_{y\to 0} \frac{y^2 y^2}{(y^2)^2 + 3y^4}$

$\lim_{y\to 0} \frac{y^4}{y^4+3y^4}$

$\lim_{y\to 0} \frac{\cancel{y^4}}{4\cancel{y^4}} = \frac{1}{4}$

set $y = 0$

$\lim_{x\to 0} \frac{0}{x^2}$

LH $\lim_{x\to 0} \frac{0}{2x}$

LH $\lim_{x\to 0} \frac{0}{2} = 0$

this limit does not exist because limit approaching from 2 different paths don't match

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5. (22 points) Consider the function $f(x, y) = \sqrt{10 - x^2 - 5y^2}$.

(a) Use a linear approximation to $f(x, y)$ at the point $(2, 1)$ to estimate the value of $f(1.95, 1.04)$.

$f(a+\Delta x, b+\Delta y) = f(a,b) + f_x(a,b)\Delta x + f_y(a,b)\Delta y$

$f(x,y) = (10 - x^2 - 5y^2)^{1/2}$ $\quad f(2,1) = 1$

$f_x(x,y) = \frac{1}{2\sqrt{10-x^2-5y^2}}(-2x)$ $\quad f_x(2,1) = \frac{1}{2}(-4) = -2$

$f_y(x,y) = \frac{1}{2\sqrt{10-x^2-5y^2}}(-10y)$ $\quad f_y(2,1) = \frac{1}{2}(-10) = -5$

$\nabla \vec{F}(2,1) = \langle -2, -5 \rangle$

$f(1.95, 1.04) \approx 1 - 2(-0.5) - 5(0.4)$

$f(1.95, 1.04) \approx 1 + 1 - 2$

$f(1.95, 1.04) \approx 0$

(b) Find the directional derivative of f at the point $(2, 1)$ in the direction of $\langle 4, 3 \rangle$.

$\nabla \vec{F}(2,1)$ found in part a as $\langle -2, -5 \rangle$

Direction $\vec{v}\ \langle 4, 3 \rangle$, but need unit vector

$\vec{u} = \left\langle \frac{4}{\sqrt{16+9}}, \frac{3}{\sqrt{16+9}} \right\rangle = \left\langle \frac{4}{5}, \frac{3}{5} \right\rangle$

$D_{\vec{u}}f = \langle -2, -5 \rangle \cdot \left\langle \frac{4}{5}, \frac{3}{5} \right\rangle$

$= -\frac{8}{5} - \frac{15}{5} = \frac{-23}{5}$

(c) Find the maximum rate of change of f at the point $(2, 1)$.

Max rate of change at $(2,1) = \|\nabla \vec{F}(2,1)\|$

$\nabla \vec{F}(2,1) = \langle -2, -5 \rangle$

$\|\nabla \vec{F}(2,1)\| = \sqrt{4+25} = \sqrt{29}$

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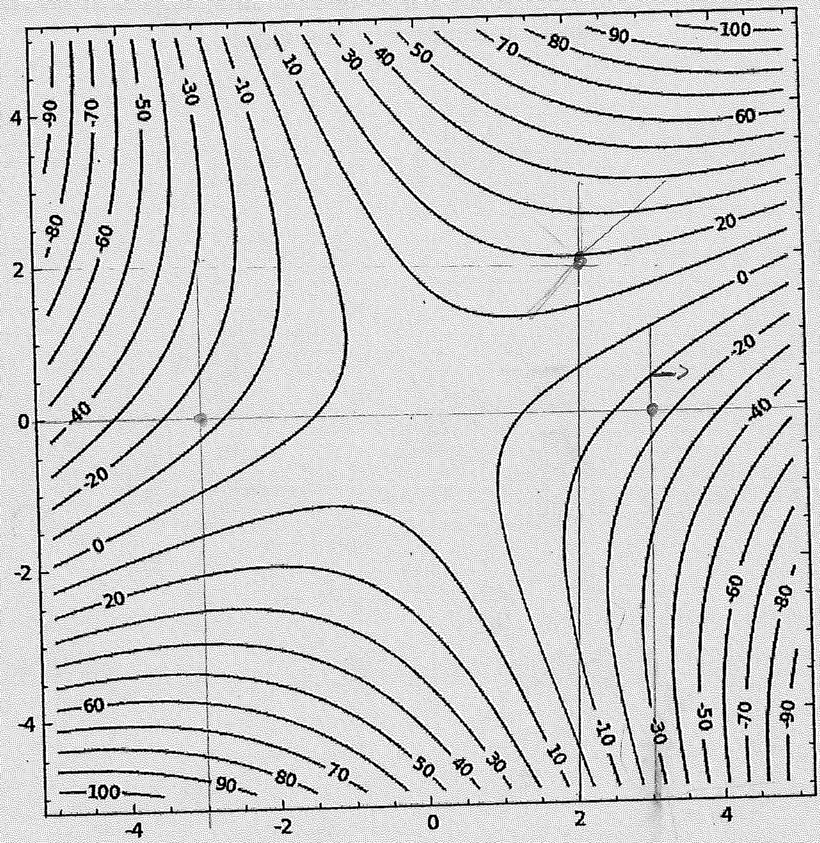
6. (20 points) Consider the contour plot for $f(x, y)$ below.



(a) Determine the sign of each of the following derivatives.

$f_x(3, 0)$ $-$ $f_y(3, 0)$ $+$

$f_{xx}(3, 0)$ $-$ decreasing at increasing rate

$f_{xy}(3, 0)$ $+$ when I go up from the point f_x is less negative

$f_{yy}(3, 0)$ $+$ hard to tell, but as I go up, it looks as though lines are getting closer so it looks like it increasing an increasing rate

(b) Give the components of a unit vector in the direction of the steepest decline at the point $(-1, 0)$. (You may estimate as necessary.)

To me, the northwest direction is the direction of steepest decline, thus this direction is $\langle -1, 1 \rangle$ but this must be a unit vector so

$\vec{u} = \langle \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

(c) Give the components of a unit vector orthogonal to $\nabla f(2, 2)$. (You may estimate as necessary.)

$\nabla f(2,2)$ is in the direction of steepest incline from $(2,2)$ which looks like it is northeast so $\nabla \vec{f}(2,2) = \langle 1, 1 \rangle$ there are 2 directions orthogonal to north east which are northwest and southeast. I will use northwest whose direction is $\langle -1, 1 \rangle$, thus the unit vector is $\vec{u} = \langle \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

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You may use this page for scratch work. Work found on this page will not be graded unless clearly indicated in the exam.