MATH 32A, Winter 2018, Midterm 2

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[redacted]IONS. EACH QUESTION IS WORTH FIVE POINTS.

NO CALCULATORS ALLOWED.

SHOW ALL YOUR WORK.

ALL UNIVERSITY AND DEPARTMENTAL POLICIES REGARDING ACADEMIC INTEGRITY APPLY.

Question:	1	2	3	4	5	Total / 25
Score:	5	5	4	5	5	24

Question 1. (5 points) $g(t) = \int_1^t \|r'(u)\| \, du$

(a) Let $\mathbf{r}(t) = \langle 4t^{1/2}, \ln t, 2t \rangle$, Compute the arc length function on the interval $[1, \infty)$, measuring from the point corresponding to $t = 1$.

(b) Given the arc length function of $\mathbf{r}(t) = \langle t, \frac{2}{3}t^{3/2}, \frac{2}{\sqrt{3}}t^{3/2} \rangle$ on $[0, \infty)$, measuring from the point corresponding to $t = 0$, is

$$g(t) = \frac{1}{6}\left((1+4t)^{3/2} - 1\right),$$

find an arc length parametrization of the same piece of curve.

a) $g(t) = \int_1^t \|r'(u)\| \, du$

$r'(t) = \langle 2t^{-1/2}, \frac{1}{t}, 2 \rangle$

$\|r'(t)\| = \sqrt{(2t^{-1/2})^2 + (\frac{1}{t})^2 + 2^2}$

$= \sqrt{4t^{-1} + t^{-2} + 4}$

$= \sqrt{(t^{-1} + 2)^2}$

$= \frac{1}{t} + 2$

$g(t) = \int_1^t \left(\frac{1}{u} + 2\right) du$

$g(t) = \ln u + 2u \Big]_1^t$

$g(t) = \ln t + 2t - \ln 1 - 2(1)$

$\boxed{g(t) = \ln t + 2t - 2}$

b) $s = g(t) = \frac{1}{6}((1+4t)^{3/2} - 1)$

$6s = (1+4t)^{3/2} - 1$

$6s + 1 = (1+4t)^{3/2}$

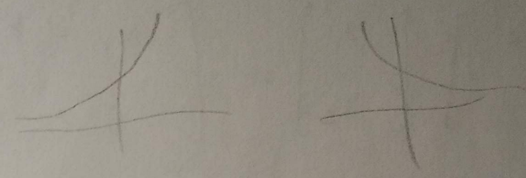
$(6s+1)^{2/3} = 1 + 4t$

$(6s+1)^{2/3} - 1 = 4t$

$t = \frac{1}{4}((6s+1)^{2/3} - 1)$ $g^{-1}(s)$

$\boxed{r_1(s) = \left\langle \frac{1}{4}((6s+1)^{2/3} - 1), \frac{2}{3}\left(\frac{1}{4}((6s+1)^{2/3} - 1)\right)^{3/2}, \frac{2}{\sqrt{3}}\left(\frac{1}{4}((6s+1)^{2/3} - 1)\right)^{3/2} \right\rangle}$

Good



Question 2. (5 points)

max

Find the value(s) of α such that the curvature of $y = e^{\alpha x}$ at $x = 0$ is as large as possible. You may use that the curvature at a point on a graph $y = f(x)$ in the plane is

$$\kappa(x) = \frac{|f''(x)|}{(1 + f'(x)^2)^{3/2}}.$$

Let $f(x) = e^{\alpha x}$

$f'(x) = \alpha e^{\alpha x}$ $\quad f'(0) = \alpha e^{\alpha(0)} = \alpha$

$f''(x) = \alpha^2 e^{\alpha x}$ $\quad f''(0) = \alpha^2 e^{\alpha(0)} = \alpha^2$

At $x=0$, $\kappa(\alpha) = \dfrac{\alpha^2}{(1+\alpha^2)^{3/2}} = \alpha^2 (1+\alpha^2)^{-3/2}$

$\kappa'(\alpha) = 2\alpha(1+\alpha^2)^{-3/2} + \alpha^2\left(-\frac{3}{2}\right)(1+\alpha^2)^{-5/2}(2\alpha)$

$= 2\alpha(1+\alpha^2)^{-5/2}\left((1+\alpha^2) + \alpha^2\left(-\frac{3}{2}\right)\right)$ $\quad \frac{2}{2} \; \frac{3}{2}$

$= \alpha(1+a^2)^{-5/2}\left(-\frac{1}{2}a^2 + 1\right)$ $\quad (+) \; (+)$

$\kappa' = 0$ when...

maximum at $\pm\sqrt{2}$

$\alpha = 0$ or $\quad -\frac{1}{2}\alpha^2 + 1 = 0$

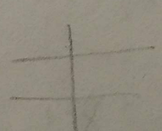
α	$-\sqrt{2}$		0		$\sqrt{2}$	
κ'	$+$	0	$-$	0	$+$	0	$-$
κ	inc		dec		inc		dec

$\downarrow$

minimum

$\frac{1}{2}\alpha^2 = 1$

$y = 1$



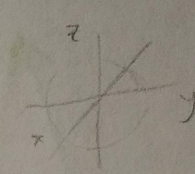
$\alpha^2 = 2$

$$\boxed{\alpha = \pm\sqrt{2}}$$

$y = e^{-\sqrt{2}x}$ & $y = e^{\sqrt{2}x}$

are reflections of each other over the y-axis, so $\kappa(x)$ is equal at $x = 0$ for ~~these~~ the graphs

Question 3. (5 points)



The functions a_T and a_N associated to $\mathbf{r}$, are determined by

$$\mathrm{a}(t) = \mathbf{r}''(t) = a_T(t)\mathbf{T}(t) + a_N(t)\mathbf{N}(t)$$

where $\mathbf{T}$, and $\mathbf{N}$ are the unit tangent, and unit normal vectors of $\mathbf{r}$, respectively.

Find $a_T(t)$ and $a_N(t)$ in the case $\mathbf{r}(t) = \langle t, \cos t, \sin t\rangle$.

$\vec{v}(t) = \vec{r}\,'(t) = \langle 1, -\sin t, \cos t \rangle$

$\vec{a}(t) = \vec{r}\,''(t) = \langle 0, -\cos t, -\sin t \rangle$

$a_T(t) = \dfrac{\vec{a} \cdot \vec{v}}{\|\vec{v}\|} = \dfrac{(1)(0) + \sin t\cos t - \sin t\cos t}{\sqrt{1^2 + \sin^2 t + \cos^2 t}} = 0$

$a_N(t) = \sqrt{\|\vec{a}(t)\|^2 - |a_T(t)|^2}$ -1

$= \sqrt{\left(\sqrt{0^2 + \cos^2 t + \sin^2 t}\right)^2 + 0^2}$

$= 1$

$$\boxed{\begin{aligned} a_T &= 0 \\ a_N &= 1 \end{aligned}}$$