MATH 32A, Winter 2018, Midterm 2

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Date: 2/23/2018

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Discussion section: 1F

THERE ARE FIVE QUESTIONS. EACH QUESTION IS WORTH FIVE POINTS.

NO CALCULATORS ALLOWED.

SHOW ALL YOUR WORK.

ALL UNIVERSITY AND DEPARTMENTAL POLICIES REGARDING ACADEMIC INTEGRITY APPLY.

Question:	1	2	3	4	5	Total / 25
Score:	5	2	5	5	5	22

Question 1. (5 points)

(a) Let $\mathbf{r}(t) = \langle 4t^{1/2}, \ln t, 2t\rangle$, Compute the arc length function on the interval $[1,\infty)$, measuring from the point corresponding to $t=1$.

(b) Given the arc length function of $\mathbf{r}(t) = \langle t, \frac{2}{3}t^{3/2}, \frac{2}{\sqrt{3}}t^{3/2}\rangle$ on $[0,\infty)$, measuring from the point corresponding to $t=0$, is

$$g(t) = \frac{1}{6}\left((1+4t)^{3/2} - 1\right),$$

find an arc length parametrization of the same piece of curve.

a) $r(t) = \langle 4t^{1/2}, \ln t, 2t \rangle$

$r'(t) = \langle 2t^{-1/2}, \frac{1}{t}, 2 \rangle$

$\|r'(t)\| = \sqrt{\left(\frac{2}{\sqrt{t}}\right)^2 + \left(\frac{1}{t}\right)^2 + (2)^2}$

$\|r'(t)\| = \sqrt{\frac{4}{t} + \frac{1}{t^2} + 4}$

$= \left(\frac{4}{t} + \frac{1}{t^2} + 4\right)^{1/2}$

$= \left(\frac{1}{t} + 2\right)^{\cancel{2}\,\cancel{1/2}}$

$t^{-2} + 4t^{-1} + 4$

$(t^{-1}+2)(t^{-1}+2)$

$(\frac{1}{t}+2)(\frac{1}{t}+2)$

$\frac{1}{t^2} + \frac{4}{t} + 4$

$\int_a^t \|r'(u)\|du = \int_1^t \left(\frac{1}{u} + 2\right) du = \ln u + 2u \Big|_1^t$

$= \ln(t) + 2t - (\cancel{\ln(1)} + 2)$

$= \boxed{\ln(t) + 2t - 2}$ ✓

b) $g(t) = \frac{1}{6}\left((1+4t)^{3/2} - 1\right)$

$6 \cdot s = \frac{1}{6}\left((1+4t)^{3/2} - 1\right) \cdot 6$

$6s = (1+4t)^{3/2} - 1$

$(6s+1)^{2/3} = (1+4t)^{\cancel{3/2}\,\cancel{2/3}}$

$(6s+1)^{2/3} = 1 + 4t$

$\frac{(6s+1)^{2/3} - 1}{4} = \frac{4t}{4}$

✓ Good

$r_1(s) = \left\langle \frac{(6s+1)^{2/3}-1}{4}, \frac{2}{3}\left[\frac{(6s+1)^{2/3}-1}{4}\right]^{3/2}, \frac{2}{\sqrt{3}}\left[\frac{(6s+1)^{2/3}-1}{4}\right]^{3/2} \right\rangle$

Question 2. (5 points)

Find the value(s) of α such that the curvature of $y = e^{\alpha x}$ at $x = 0$ is as large as possible. You may use that the curvature at a point on a graph $y = f(x)$ in the plane is

$$\kappa(x) = \frac{|f''(x)|}{(1 + f'(x)^2)^{3/2}}.$$

$y = e^{\alpha x}$

$y' = \alpha e^{\alpha x}$

$y'' = \alpha^2 e^{\alpha x}$

$K(x) = \dfrac{|\alpha^2 e^{\alpha x}|}{(1 + (\alpha e^{\alpha x})^2)^{3/2}}$ $\quad (1 + \alpha^2 e^{2\alpha x})^{3/2}$

3

$K'(x) = \dfrac{\alpha^3 e^{\alpha x}(1 + (\alpha e^{\alpha x})^2)^{3/2} - \frac{3}{2}(1 + (\alpha e^{\alpha x})^2)^{1/2}(2\alpha e^{\alpha x})(\alpha^2 e^{\alpha x})}{(1 + (\alpha e^{\alpha x})^2)^3}$ $\quad 2\alpha^3 e^{2\alpha x}$

$K'(x) = (\alpha^3 e^{\alpha x})'(1 + (\alpha e^{\alpha x})^2)^{1/2}((1 + (\alpha e^{\alpha x})^2) - 3) = 0$

$(\alpha^3)(1 + \alpha^2)^{1/2}((1 + \alpha^2) - 3) = 0$

$1 + \alpha^2 = 0$ $\qquad 1 + \alpha^2 = 3$

$\alpha^2 = -1$ $\qquad \alpha^2 = 2$

$\boxed{\alpha = 0, \quad \alpha = \pm\sqrt{2}}$

$T = \frac{r'(t)}{\|r'(t)\|}$

$a_T = a \cdot T$

Question 3. (5 points)

The functions $a_\mathbf{T}$ and $a_\mathbf{N}$ associated to $\mathbf{r}$, are determined by

$$\mathbf{a}(t) = \mathbf{r}''(t) = a_\mathbf{T}(t)\mathbf{T}(t) + a_\mathbf{N}(t)\mathbf{N}(t)$$

where $\mathbf{T}$, and $\mathbf{N}$ are the unit tangent, and unit normal vectors of $\mathbf{r}$, respectively.

Find $a_\mathbf{T}(t)$ and $a_\mathbf{N}(t)$ in the case $\mathbf{r}(t) = \langle t, \cos t, \sin t\rangle$.

$r(t) = \langle t, \cos t, \sin t \rangle$

$r'(t) = \langle 1, -\sin t, \cos t \rangle$

$r''(t) = \langle 0, -\cos t, -\sin t \rangle$

$\sqrt{1 + \sin^2 t + \cos^2 t} = \sqrt{2}$

$T = \frac{r'(t)}{\|r'(t)\|} = \frac{\langle 1, -\sin t, \cos t \rangle}{\sqrt{2}} = \left\langle \frac{1}{\sqrt{2}}, \frac{-\sin t}{\sqrt{2}}, \frac{\cos t}{\sqrt{2}} \right\rangle$

$a_T = a \cdot T = \langle 0, -\cos t, -\sin t \rangle \cdot \left\langle \frac{1}{\sqrt{2}}, \frac{-\sin t}{\sqrt{2}}, \frac{\cos t}{\sqrt{2}} \right\rangle$

$= 0 + \frac{\cos t \sin t}{\sqrt{2}} - \frac{\cos t \sin t}{\sqrt{2}} = \boxed{0}$

$a_N N = a - a_T T = \langle 0, -\cos t, -\sin t \rangle$

$a_N = \|a_N N\| = \sqrt{0 + \cos^2 t + \sin^2 t} = \boxed{1}$

$N = \frac{a_N N}{a_N} = \langle 0, -\cos t, -\sin t \rangle$

$a = N$ where T is $\left\langle \frac{1}{\sqrt{2}}, \frac{-\sin t}{\sqrt{2}}, \frac{\cos t}{\sqrt{2}} \right\rangle$
and N is $\langle 0, -\cos t, -\sin t \rangle$

$\boxed{a_T(t) = 0 \qquad a_N(t) = 1}$

Question 4. (5 points)

Evaluate the limit or determine that it does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}.$$

$x = r\cos\theta$
$y = r\sin\theta$

$$\lim_{r\to 0} \frac{(r\cos\theta)(r\sin\theta)}{\sqrt{(r\cos\theta)^2+(r\sin\theta)^2}}$$

$$\lim_{r\to 0} \frac{r^2\cos\theta\sin\theta}{\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}}$$

$$\lim_{r\to 0} \frac{r^{\cancel{2}}\cos\theta\sin\theta}{\cancel{r}\sqrt{\cos^2\theta+\sin^2\theta}} \to 1$$

$$\lim_{r\to 0} r\,\underbrace{\cos\theta\sin\theta}_{\text{bounded}} = \boxed{0}$$

$$\frac{mx^2}{\sqrt{x^2+m^2x^2}} \qquad \frac{m\cancel{x^2}}{\cancel{x}\sqrt{1-m^2}}\ x$$

Question 5. (5 points)

Consider the functions,

A. $g(x, y) = \cos(x^2 + y^2)$,

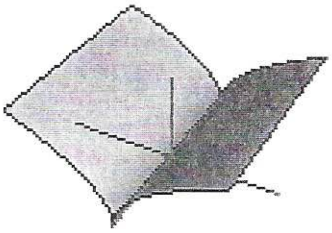
B. $g(x, y) = x + y^2$,

C. $g(x, y) = (x^2 + y^2)^{1/4}$,

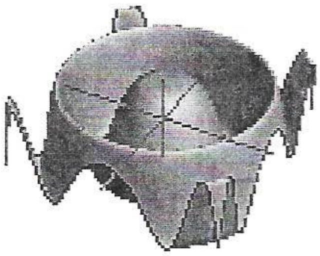
D. $g(x, y) = |x|^{1/2}$,

E. $g(x, y) = \cos(x)\cos(y)$.

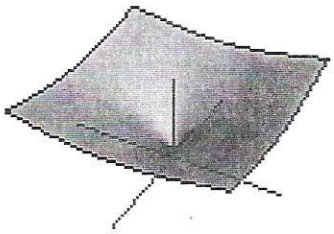
Match each to the correct graph by writing the corresponding letter under the graph.



D



A



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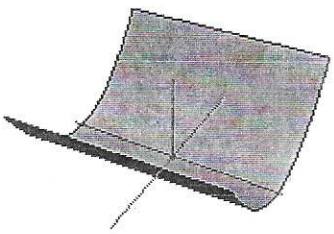
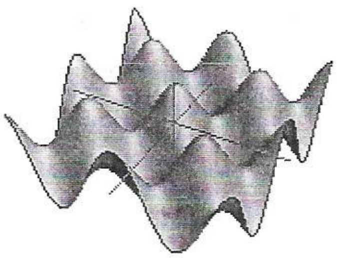


B



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