MATH 32A, Winter 2018, Midterm 1

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Discussion section: 1F

THERE ARE EIGHT QUESTIONS. EACH QUESTION IS WORTH FIVE POINTS.

NO CALCULATORS ALLOWED.

SHOW ALL YOUR WORK.

ALL UNIVERSITY AND DEPARTMENTAL POLICIES REGARDING ACADEMIC INTEGRITY APPLY.

Question:	1	2	3	4	5	6	7	8	Total / 40
Score:	5	5	5	5	5	5	5	5	40

Question 1. (5 points)

Find the projection of **u** along **v**, with $\mathbf{u} = \langle 1, -2, 1\rangle$, and $\mathbf{v} = \langle 0, 3, -1\rangle$.

$$\vec{u}_{\parallel \vec{v}} = \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}$$

$$\vec{u}\cdot\vec{v} = 1(0) - 2(3) + 1(-1)$$
$$= 0 - 6 - 1$$
$$= -7$$

$$\vec{v}\cdot\vec{v} = \|\vec{v}\|^2 = 0(0) + 3(3) - 1(-1)$$
$$= 0 + 9 + 1$$
$$= 10$$

$$\boxed{\vec{u}_{\parallel \vec{v}} = \left(\frac{-7}{10}\right)\langle 0, 3, -1\rangle}$$

$$\boxed{\vec{u}_{\parallel \vec{v}} = \left\langle 0, \frac{-21}{10}, \frac{7}{10}\right\rangle}$$

same length

Question 2. (5 points)

Let **v**, **w** be non-zero vectors such that $\|\mathbf{v}\| = \|\mathbf{w}\|$. Use the dot product to show that $\mathbf{v}+\mathbf{w}$ and $\mathbf{v}-\mathbf{w}$ are orthogonal.

dot product $= \|\vec{v}\|\|\vec{w}\|\cos\theta \rightarrow$ must $= 0$ if vectors are orthogonal

$$(v+w)\cdot(v+w) = \|v+w\|^2$$

$$v\cdot v + 2(v\cdot w) + w\cdot w = \|\vec{v}+w\|^2$$

$$\|v\|^2 + 2(v\cdot w) + \|w\|^2 = \|v+w\|^2$$

let $\|v\| = \|w\| = r$

$$r^2 + 2(v\cdot w) + r^2 = (2r)^2$$

$$\cancel{2r^2} + 2(v\cdot w) = 4r^2$$
$$-\cancel{2r^2} \qquad -2r^2$$

$$\cancel{2}(v\cdot w) = \cancel{2}r^2$$

$$(v\cdot w) = r^2$$

$$(v+w)\cdot(v-w) = 0$$

$-1 = \lambda$ scalar

$$v\cdot v + v\cdot w + v\cdot -w + -w\cdot w = 0$$

$$\cancel{\|v\|^2} + v\cdot w - (v\cdot w) + -\cancel{\|w\|^2} = 0$$

since $\|v\| = \|w\|$

$$v\cdot w - v\cdot w = 0$$

$$0 = 0 \checkmark$$

$$v\cdot v + v\cdot w + v\cdot -w + w\cdot -w = 0$$

$$\cancel{\|v\|^2} + (v\cdot w) + -(v\cdot w) - \cancel{\|w\|^2} = 0$$

$$0 = 0$$

So $(\vec{v}+\vec{w}) \perp (\vec{v}-\vec{w})$ ∴ orthogonal

$\hookrightarrow \|v+w\| = 2v$

$\|v-w\| = 0$

dot product $\|v+w\|\,\|v-w\|\cos\theta$

$\|2v\|(0)\cos\theta = 0$

so ~~always~~ orthogonal ✓

Question 3. (5 points)

Consider lines

$$\mathcal{L}_1, \text{ with parametrization } \mathbf{r}_1(s) = s\,\langle 1,1,1\rangle\,,$$

and

$$\mathcal{L}_2, \text{ with parametrization } \mathbf{r}_2(t) = \langle 1,2,3\rangle + t\,\langle 2,1,0\rangle\,.$$

Do they intersect? If so, where?

$\vec{r}_1(s) = \langle s, s, s\rangle$

$\vec{r}_2(t) = \langle 1+2t, 2+t, 3\rangle$

if they intersect, there must be s + t values such that $r_1(s) = r_2(t)$

$s = 1+2t$ $\quad$ $s = 2+t$ $\quad$ $\underline{s = 3}$

$3 = 1+2(1)$ $\quad$ $3 = 2+t$

$3 = 3$ ✓ $\quad$ $\underline{1 = t}$

intersect

point of intersection: $\vec{r}_1(3) = \langle 3,3,3\rangle$

$\vec{r}_2(1) = \langle 3,3,3\rangle$

Yes, $\mathcal{L}_1$ and $\mathcal{L}_2$ intersect at point $\boxed{\langle 3,3,3\rangle}$ where $t=1$ and $s=3$.

Question 4. (5 points)

Calculate the area of the parallelogram spanned by $\mathbf{u} = \langle -2, 0, 1\rangle$, and $\mathbf{v} = \langle 1, 0, -3\rangle$.

$\wedge$ = unit vectors

area $= \|\vec{u} \times \vec{v}\|$ ★ magnitude

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 0 & 1 \\ 1 & 0 & -3 \end{vmatrix} = \hat{i}\left(-3(0) - 0(1)\right) - \hat{j}\left(-2(-3) - 1(1)\right) + \hat{k}\left(-2(0) - 0(1)\right)$$

$$= \hat{i}(0) - \hat{j}(6-1) + \hat{k}(0) = \langle 0,-5,0\rangle$$

$$\cancel{= \hat{i}(-3) - \hat{j}(6-1) + \hat{k}(0)} \quad 0\hat{i} - 5\hat{j} + 0k$$

$$\cancel{= -3\hat{i} - 5\hat{j} = \langle -3,-5,0\rangle}$$

$$\cancel{\|\vec{u} \times \vec{v}\| = \sqrt{(-3)^2 + (-5)^2 + 0^2}}$$

$$\cancel{= \sqrt{9 + 25 + 0}}$$

$$\cancel{= \sqrt{34}}$$

$$\|\vec{u} \times \vec{v}\| = \sqrt{0^2 + (-5)^2 + 0^2}$$

$$= \sqrt{0 + 25 + 0}$$

$$= \sqrt{25} = 5$$

$\boxed{\text{area} = 5}$

√ cross product since $\vec{u} \times \vec{v}$ is orthogonal to $\vec{u}$ + $\vec{v}$, the dot products should be zero.

$$-2(-3) + 0(-5) + 1(0) = 6 \neq 0$$

√ again

$$u \cdot (u \times v) = -2(0) + 0(-5) + 1(0) = 0$$

$$0 = 0 \checkmark$$

$$v \cdot (u \times v) = 1(0) + 0(-5) + -3(0) = 0$$

$$0 = 0 \checkmark$$

ok $\frac{5}{5}$

Question 5. (5 points)

$u = \langle x, y, z \rangle$

Let $P = (1,1,1)$, $Q = (2,0,0)$, and $R = (0,2,0)$. Identify vectors $\mathbf{n}$ and $\mathbf{u}_0$ such that $\mathbf{n} \cdot (\mathbf{u} - \mathbf{u}_0) = 0$ is a vector equation for the plane containing P, Q, and R.

vector equation $= \vec{n} \cdot \vec{OP} = 0$

$\vec{PQ} = \langle 1, -1, -1 \rangle$

$\vec{PR} = \langle -1, 1, -1 \rangle$

$$\vec{n} = \vec{PQ} \times \vec{PR}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & -1 \\ -1 & 1 & -1 \end{vmatrix}$$

$$= \hat{i}\big(\overset{1}{-1(-1)} - \overset{--1}{1(-1)}\big) - \hat{j}\big(\underset{-1}{1(-1)} - \underset{-1}{-1(-1)}\big) + \hat{k}\big(\underset{1}{1(1)} - \underset{-1}{-1(-1)}\big)$$

$$= \hat{i}(2) - \hat{j}(-2) + \hat{k}(0)$$

$$= 2\hat{i} + 2\hat{j} + 0\hat{k}$$

$$\boxed{\vec{n} = \langle 2, 2, 0 \rangle}$$

$\vec{u}_0 = -\vec{OP}$

$= -\langle 1, 1, 1 \rangle$

~~$\vec{u}_0 = \langle -1, -1, -1 \rangle$~~

$u_0 =$

vector equation =

$\langle 2, 2, 0 \rangle \cdot \langle \vec{u} - \langle 1, 1, 1 \rangle \rangle = 0$

using another point in plane

$2x + 2y + 0z = d$

using point P

$2(1) + 2(1) + 0(1) = d$

$2 + 2 = d$

$4 = d$

$\underbrace{2x + 2y + 0z = 4}$

~~vector equation~~ plane equation

$$\boxed{\vec{u}_0 = \langle 1, 1, 1 \rangle}$$

$\vec{u} = \langle x, y, z \rangle = \boxed{\langle 2, 0, 0 \rangle \text{ point } Q}$

$\checkmark$ ~~$\langle 2, 2, 0 \rangle \cdot \langle -1, -1, -1 \rangle = 0$~~

~~$2(-1) + 2(-1) + 0 = 0$~~

$\langle 2, 2, 0 \rangle \cdot \langle 1, -1, -1 \rangle$

$2(1) + 2(-1) + 0 = 0$

$2 - 2 = 0$ ✓

$\sin^2 + \cos^2 = 1$

Question 6. (5 points)

Use parameter θ, and two vector valued functions, $\mathbf{r}_+(\theta)$, $\mathbf{r}_-(\theta)$, to parametrize the intersection of $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$.

$y^2 = 1 - x^2 \quad z^2 = 1 - x^2$

$x^2 + y^2 = 1$

$x^2 + z^2 = 1$

let

$x = \cos\theta$

$y = \sin\theta$

$z^2 = 1 - \cos^2\theta$

$\cos^2\theta + \sin^2\theta = 1$ ✓

$\sqrt{z^2} = \sqrt{\sin^2\theta}$

$\cos^2\theta + \sin^2\theta = 1$ ✓

$z = \pm \sin\theta$

$$\boxed{\begin{aligned} \vec{r}_+(\theta) &= \langle \cos\theta, \sin\theta, +\sin\theta \rangle \\ \vec{r}_-(\theta) &= \langle \cos\theta, \sin\theta, -\sin\theta \rangle \end{aligned}}$$

✓

double check

Question 7. (5 points)

Let $\mathbf{r}(s) = \langle s^2 - 1, s \rangle$, and $g(t) = \sin(t)$. Consider curve C with parametrization $\mathbf{r}(g(t))$. Find a parametrization of the tangent line to C at the point corresponding $\boxed{\text{to } t = 0.}$

tangent line $C = \vec{r}(s) + t\, r'(s)$

$\vec{r}(s) = \langle s^2 - 1, s \rangle$

$\sin^2\theta + \cos^2\theta = 1$

$g(t) = \sin t$

$\sin(0) = 0$

$r(g(t))$?

$\vec{r}(\sin t) = \langle \sin^2 t - 1, \sin t \rangle$

$\vec{r}(\sin 0) = \langle \sin^2(0) - 1, \sin 0 \rangle$

$\vec{r}(0) = \langle -1, 0 \rangle \rightarrow$ point

$\rightarrow$ chain rule =

$g'(t)\, r'(s)$

$\cos t \langle 2s, 1 \rangle$

$\cos t \langle 2\sin t, \cancel{\sin t} \rangle$

$\vec{r}\,'(\sin t) = \langle 2 \sin t \cos t, \cos t \rangle$

Let tangent line to C at point $t=0$ be $T(t)$

$f(g(x)) = f'(g(x)) \cdot g'(x)$

$\boxed{T(t) = \langle -1, 0 \rangle + t \langle 0, 1 \rangle}$

$r'(\sin 0) = \langle 2 \sin 0 \cos 0, \cos 0 \rangle$

$= \langle 2(0)(1), 1 \rangle$

$r'(0) = \langle 0, 1 \rangle$

$T(t) = \langle -1, t \rangle$

$f(x) = s^2 - 1$

$g(x) = \sin x \rightarrow \sin^2 x - 1$

$f'(g(x)) = 2\sin x \cos x$

$f(x) = \cancel{x}$

$g(x) = \sin x$

$f(g(x)) = \sin x$

$f'(g(x)) = \cos x$

Question 8. (5 points)

Suppose a particle moves in space with position vector $\mathbf{r}(t)$, and velocity vector

$$\mathbf{r}'(t) = \left\langle 2t^{-\frac{1}{2}}, 1, 2t \right\rangle.$$

initial condition

If $\mathbf{r}(1) = \langle 4, 1, 1 \rangle$, what is the position of the particle at time $t = 4$?

integrate

$$\int \vec{r}'(t) = \int \langle 2t^{-1/2}, 1, 2t \rangle$$

$2t^{-1/2} \quad -1/2 + 1 = 1/2$

$\frac{2}{1/2} = 4$

$\cancel{4} \; 4t^{1/2}$

$$\vec{r}(t) = \left\langle \int 2t^{-1/2} dt, \int 1 \, dt, \int 2t \, dt \right\rangle$$

$$\downarrow$$

$$\vec{r}(t) = \langle 4t^{1/2}, t, t^2 \rangle + C$$

$$\vec{r}(1) = \langle 4(1)^{1/2}, 1, 1^2 \rangle + C = \langle 4, 1, 1 \rangle$$

$$\langle 4, 1, 1 \rangle + C = \langle 4, 1, 1 \rangle$$

$$C = \langle 0, 0, 0 \rangle$$

$$\vec{r}(4) = \langle 4(4)^{1/2}, 4, 4^2 \rangle + \langle 0, 0, 0 \rangle$$

$$\vec{r}(4) = \langle 4\sqrt{4}, 4, 16 \rangle$$

$\boxed{\vec{r}(4) = \langle 8, 4, 16 \rangle}$ position at $t = 4$

$$\vec{r}(4) = 4\langle 2, 1, 4 \rangle$$