MATH 32A, Winter 2018, Midterm 1

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Date: 2/2/2018

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Discussion section: [redacted]

THERE ARE EIGHT QUESTIONS. EACH QUESTION IS WORTH FIVE POINTS.

NO CALCULATORS ALLOWED.

SHOW ALL YOUR WORK.

ALL UNIVERSITY AND DEPARTMENTAL POLICIES REGARDING ACADEMIC INTEGRITY APPLY.

Question:	1	2	3	4	5	6	7	8	Total / 40
Score:	5	5	5	5	2	0	5	5	32

Question 1. (5 points)

Find the projection of $\mathbf{u}$ along $\mathbf{v}$, with $\mathbf{u} = \langle 1, -2, 1 \rangle$, and $\mathbf{v} = \langle 0, 3, -1 \rangle$.

$$u_{\parallel v} = \left(\frac{u \cdot v}{v \cdot v}\right) v = \left(-\frac{7}{10}\right) v = \left\langle \frac{-7}{10}(0), \frac{-7}{10}(3), \frac{-7}{10}(-1) \right\rangle$$

$$u_{\parallel v} = \boxed{\left\langle 0, \frac{-21}{10}, \frac{7}{10} \right\rangle}$$

$$u \cdot v = (1)(0) + (-2)(3) + (1)(-1)$$
$$= 0 - 6 - 1 = -7$$

$$v \cdot v = (0)(0) + (3)(3) + (-1)(-1)$$
$$= 9 + 1 = 10$$

Question 2. (5 points)

Let $\mathbf{v}$, $\mathbf{w}$ be non-zero vectors such that $\|\mathbf{v}\| = \|\mathbf{w}\|$. Use the dot product to show that $\mathbf{v}+\mathbf{w}$ and $\mathbf{v}-\mathbf{w}$ are orthogonal.

$$(v+w)\cdot(v-w) = \|v\|^2 + (w\cdot v) - (v\cdot w) - \|w\|^2$$

$$= \|v\|^2 + \cancel{(w\cdot v) - (v\cdot w)} - \|w\|^2$$

$$= \|v\|^2 - \|w\|^2 = 0, \text{ since } \|v\| = \|w\|$$

since $(v+w)\cdot(v-w) = 0$, they are orthogonal.

Question 3. (5 points)

Consider lines

$$\mathcal{L}_1, \text{ with parametrization } \mathbf{r}_1(s) = s\,\langle 1, 1, 1 \rangle\,,$$

and

$$\mathcal{L}_2, \text{ with parametrization } \mathbf{r}_2(t) = \langle 1, 2, 3 \rangle + t\,\langle 2, 1, 0 \rangle\,.$$

Do they intersect? If so, where?

$x: \quad 1s = 1 + 2t$

$y: \quad 1s = 2 + t$

$z: \quad 1s = 3$

$$\begin{array}{rcl} s &=& 1 + 2t \\ -s &=& -2 - t \\ \hline 0 &=& -1 + t \\ t &=& 1 \end{array}$$

$s = 1 + 2(1) = 3$

$s = 2 + 1 = 3$

$s = 3$

$r_1(3) = 3 <1, 1, 1> = <3, 3, 3>$

$r_2(1) = <1, 2, 3> + <2, 1, 0> = <3, 3, 3>$

$\boxed{\text{yes, they intersect at } <3, 3, 3>}$

Question 4. (5 points)

Calculate the area of the parallelogram spanned by $\mathbf{u} = \langle -2, 0, 1 \rangle$, and $\mathbf{v} = \langle 1, 0, -3 \rangle$.

$$A_{\text{parallelogram}} = \| u \times v \|$$

$$\begin{array}{ccc} i & j & k \\ \hline \end{array}$$
$$\begin{vmatrix} -2 & 0 & 1 \\ 1 & 0 & -3 \end{vmatrix}$$

$$u \times v = (0-0)i - (6-1)j + (0-0)k$$

$$= -5j \qquad \sqrt{5^2} = 5$$

$$\| u \times v \| = \boxed{5 \text{ sq. units}}$$

Question 5. (5 points)

Let $P = (1,1,1)$, $Q = (2,0,0)$, and $R = (0,2,0)$. Identify vectors $\mathbf{n}$ and $\mathbf{u}_0$ such that $\mathbf{n} \cdot (\mathbf{u} - \mathbf{u}_0) = 0$ is a vector equation for the plane containing P, Q, and R.

$$\overrightarrow{PQ} = \langle 1, 1, 1 \rangle + t \langle \overset{2-1}{1}, \overset{0-1}{-1}, \overset{0-1}{-1} \rangle$$

$$\overrightarrow{PR} = \langle 1, 1, 1 \rangle + t \langle \overset{0-1}{-1}, \overset{2-1}{1}, \overset{0-1}{-1} \rangle$$

$$n = \left| \overrightarrow{PQ} \times \overrightarrow{PR} \right| \begin{array}{ccc} i & j & k \\ \end{array} \left| \begin{array}{ccc} 1 & -1 & -1 \\ -1 & 1 & -1 \end{array} \right|$$

$$n = (1 - (-1))i - (-1 - (1))j + (1 - 1)k$$

$$n = 2i + 2j + k$$

$$\boxed{\begin{array}{l} n = \langle 2, 2, \cancel{1}\,0 \rangle \\ u_0 = \langle 1, 1, 1 \rangle \end{array}}$$

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Question 6. (5 points)

Use parameter θ, and two vector valued functions, $\mathbf{r}_+(\theta)$, $\mathbf{r}_-(\theta)$, to parametrize the intersection of $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$.

$\cos^2\theta + \sin^2\theta = 1$ $\quad\quad$ $\cos^2\theta + \sin^2\theta = 1$

$\cancel{\langle \cos\theta, \sin\theta, \sin\theta \rangle}$

$\langle \pm\cos\theta, \sin\theta, 0 \rangle$

$\langle \pm\cos\theta, 0, \sin\theta \rangle$

$x: \quad \cos\theta = \cos\theta \quad\quad -\cos\theta = -\cos\theta$

$y: \quad \sin\theta = 0$

$z = \quad 0 = \sin\theta$

$\theta = 0$

$r_+(\theta) = \langle 1, 0, 0 \rangle + t\langle \cos\theta, \sin\theta, \sin\theta \rangle$

$r_-(\theta) = \langle -1, 0, 0 \rangle + t\langle -\cos\theta, \sin\theta, \sin\theta \rangle$

$0 \leq \theta \leq 2\pi$

Question 7. (5 points)

Let $\mathbf{r}(s) = \langle s^2 - 1, s\rangle$, and $g(t) = \sin(t)$. Consider curve $\mathcal{C}$ with parametrization $\mathbf{r}(g(t))$. Find a parametrization of the tangent line to $\mathcal{C}$ at the point corresponding to $t = 0$.

$r'(g(t)) = g'(t)\, r'(g(t))$

$g'(t) = \cos(t)$

$r'(s) = \langle 2s, 1\rangle$

$r'(g(t)) = \langle 2\sin t, 1\rangle$

$\cos t \langle 2\sin t, 1\rangle$

$r'(g(t)) \langle 2\cos t \sin t, \cos t\rangle$

$r(g(t)) = \sin^2 t - 1, \sin t$

$\cancel{r'(g(t))}$

$L(t) = r(g(t)) + t\,(r'(g(t)))$

$L(t) = \boxed{\langle -1, 0\rangle + t\langle 0, 1\rangle}$

$r'(g(0)) = \langle 0, 1\rangle$

$r(g(0)) = \langle -1, 0\rangle$

Question 8. (5 points)

Suppose a particle moves in space with position vector $\mathbf{r}(t)$, and velocity vector

$$\mathbf{r}'(t) = \left\langle 2t^{-\frac{1}{2}}, 1, 2t \right\rangle.$$

If $\mathbf{r}(1) = \langle 4, 1, 1 \rangle$, what is the position of the particle at time $t = 4$?

$\int r'(t) = \int < 2t^{-1/2}, 1, 2t >$

$r(t) = < 4t^{1/2}, t, t^2 > + < C_1, C_2, C_3 >$

$<4, 1, 1> = r(1) = < 4(1)^{1/2}, 1, 1 > + < C_1, C_2, C_3 >$

$< 4, 1, 1 > = < 4, 1, 1 > + < C_1, C_2, C_3 >$

$< C_1, C_2, C_3 > = < 0, 0, 0 >$

$r(4) = < 4(4)^{1/2}, 4, 4^2 >$

$r(4) = \boxed{< 8, 4, 16 >}$