Math 32A - Midterm 2

Name:

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Problems	Points	Score
1	30	30
2	30	30
3	25	25
4	15	9
Total	100	94

Problem 1. *(30 points)* Consider the curve given by

$$\mathbf{r} : [0, +\infty) \to \mathbb{R}^3, \quad \mathbf{r}(t) = \left\langle \sin\left(\frac{t^2}{2}\right), \cos\left(\frac{t^2}{2}\right), t^2 \right\rangle.$$

(a) *(20 points)* Find the arc length parametrization of the above curve.
(b) *(10 points)* Compute the curvature of the above curve.
Hint: You can use part (a).

a. $r'(t) = \langle t\cos\left(\frac{t^2}{2}\right), -t\sin\left(\frac{t^2}{2}\right), 2t \rangle$

$\|r'(t)\| = \sqrt{\left(t\cos\left(\frac{t^2}{2}\right)\right)^2 + \left(t\sin\left(\frac{t^2}{2}\right)\right)^2 + (2t)^2}$

$= \sqrt{t^2\left(\cos^2\left(\frac{t^2}{2}\right) + \sin^2\left(\frac{t^2}{2}\right)\right) + 4t^2}$

$= \sqrt{t^2 + 4t^2} = \sqrt{5t^2} = \sqrt{5}\,t$

$s = \int_0^t \sqrt{5}\,t'\,dt' = \frac{\sqrt{5}}{2}t^2$

$t^2 = \frac{2}{\sqrt{5}}s$

$\Rightarrow t = \sqrt{\frac{2s}{\sqrt{5}}}$

$r(s) = \left\langle \sin\left(\frac{\frac{2s}{\sqrt{5}}}{2}\right), \cos\left(\frac{\frac{2s}{\sqrt{5}}}{2}\right), \frac{2s}{\sqrt{5}} \right\rangle$

$\boxed{r(s) = \left\langle \sin\left(\frac{s}{\sqrt{5}}\right), \cos\left(\frac{s}{\sqrt{5}}\right), \frac{2s}{\sqrt{5}} \right\rangle}$

b. $T(s) = \left\langle \frac{1}{\sqrt{5}}\cos\left(\frac{s}{\sqrt{5}}\right), \frac{-1}{\sqrt{5}}\sin\left(\frac{s}{\sqrt{5}}\right), \frac{2}{\sqrt{5}} \right\rangle$ $\qquad T(s) = \frac{r'(s)}{\|r'(s)\|}$, $\|r'(s)\| = 1$

$T'(s) = \left\langle -\frac{1}{5}\sin\left(\frac{s}{\sqrt{5}}\right), \frac{-1}{5}\cos\left(\frac{s}{\sqrt{5}}\right), 0 \right\rangle$

$\kappa(s) = \|T'(s)\| = \sqrt{\frac{1}{25}\sin^2\left(\frac{s}{\sqrt{5}}\right) + \frac{1}{25}\cos^2\left(\frac{s}{\sqrt{5}}\right)} = \sqrt{\frac{1}{25}} = \boxed{\frac{1}{5} = \kappa(s)}$

Problem 2. *(30 points)* Consider a curve whose acceleration is given by

$$\mathbf{a}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} \text{ for all } t,$$

and for which we know that the velocity and the position vector at time $t = \frac{\pi}{2}$ are

$$\mathbf{v}\left(\frac{\pi}{2}\right) = 3\mathbf{k} \text{ and } \mathbf{r}\left(\frac{\pi}{2}\right) = 3\mathbf{i},$$

where

$$\mathbf{i} = \langle 1,0,0\rangle, \quad \mathbf{j} = \langle 0,1,0\rangle, \quad \mathbf{k} = \langle 0,0,1\rangle.$$

(a) *(20 points)* Find **r** for all t.

(b) *(10 points)* Find the tangential and the normal components of the acceleration vector at any time t.

a. $v(t) = \int a(t)\,dt = \sin(t)i - \cos(t)j + c_1$

$v(\frac{\pi}{2}) = \sin(\frac{\pi}{2})i - \cos(\frac{\pi}{2})j + c_1$

$= i + c_1 = 3k$

$\Rightarrow c_1 = -i + 3k$

$v(t) = (\sin(t) - 1)i - \cos(t)j + 3k$

$r(t) = \int v(t)\,dt = (-\cos t - t)i - \sin(t)j + 3tk + c_2$

$r(\frac{\pi}{2}) = (-\cos(\frac{\pi}{2}) - \frac{\pi}{2})i - \sin(\frac{\pi}{2})j + \frac{3\pi}{2}k + c_2 = 3i$

$= -\frac{\pi}{2}i - j + \frac{3\pi}{2}k + c_2 = 3i$

$\Rightarrow c_2 = (3 + \frac{\pi}{2})i + j - \frac{3\pi}{2}k$

$r(t) = (-\cos t - t)i - \sin(t)j + 3tk + (3 + \frac{\pi}{2})i + j - \frac{3\pi}{2}k$

$\boxed{r(t) = (-\cos t - t + (3 + \frac{\pi}{2}))i + (-\sin t + 1)j + (3t - \frac{3\pi}{2})k}$

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(Part B on back)

. $a(t) = a_N N + a_T T$ $\quad v(t) = \|v\| = \sqrt{\sin^2 t - 2\sin t + 1 + \cos^2 t + 9} = \sqrt{-2\sin t + 11}$

$a_T(t) = v'(t) = \frac{1}{2}(-2\sin t + 1\,)^{-1/2}(-2\cos t) = \boxed{\frac{-\cos t}{\sqrt{-2\sin t + 11}} = a_T(t)}$

$a_N(t) = \sqrt{\|a(t)\|^2 - |a_T(t)|^2} = \sqrt{1 - \left(\frac{-\cos t}{\sqrt{-2\sin t + 11}}\right)^2}$

$\|a(t)\| = \sqrt{\cos^2 t + \sin^2 t} = 1$

$a_N(t) = \sqrt{1 - \frac{\cos^2 t}{-2\sin t + 11}} = \boxed{\sqrt{1 + \frac{\cos^2 t}{\sin t + 11}} = a_N(t)}$

$a(t) = \frac{-\cos t}{\sqrt{-2\sin t + 11}} T + \sqrt{1 + \frac{\cos^2 t}{\sin t + 11}} N$

Problem 3. *(25 points)*

(a) *(15 points)* Compute the limit

$$\lim_{(x,y)\to(0,3)} xy^3 \ln(x).$$

(b) *(10 points)* Let

$$\mathbf{r} : \mathbb{R} \to \mathbb{R}^2, \quad \mathbf{r}(t) = \langle f(t), g(t)\rangle,$$

be a twice differentiable vector valued function representing a curve in arc length parametrization whose curvature is never zero. Show that its curvature is given by the formula

$$\kappa(t) = |f'(t)g''(t) - f''(t)g'(t)| \text{ for all } t.$$

Hint: Consider **r** *as a curve in* $\mathbb{R}^3$ *and use the formula for* κ *that involves a cross product.*

a. $\left(\lim_{x\to 0} x\ln x\right)\left(\lim_{y\to 3} y^3\right)$

$\lim_{y\to 3} y^3 = 3^3 = 27$

$\lim_{x\to 0} \frac{\ln x}{\frac{1}{x}} = \frac{\ln(0)}{\frac{1}{0}} = \frac{\infty}{\infty}$

L'Hopital

$\lim_{x\to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0} \frac{-x^2}{x} = \lim_{x\to 0} -x = 0$

$0 \cdot 27 = 0$, $\therefore \boxed{\lim_{(x,y)\to(0,3)} xy^3 \ln x = 0}$

b. $\kappa(t) = \frac{\|r' \times r''\|}{\|r'\|^3}$

$r'(t) = \langle f'(t), g'(t)\rangle$

$r''(t) = \langle f''(t), g''(t)\rangle$

$r'(t) \times r''(t) = \begin{vmatrix} f'(t) & g'(t) \\ f''(t) & g''(t) \end{vmatrix} = f'(t)g''(t) - f''(t)g'(t)$

$\|r'(t)\|$ in arc length par. $= 1$

$\boxed{\therefore \quad \kappa(t) = |f'(t)g''(t) - f''(t)g'(t)| \ \forall t}$

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Problem 4. *(15 points)* Let **r** be a vector valued function representing a curve in arc length parametrization whose curvature is never zero and whose Frenet frame is given by $\{T, N, B\}$. Assume that

$$\sin(2t)T(t) + \cos(2t)N(t) + B(t) = c \text{ for all } t,$$

where c is a non-zero vector in $\mathbb{R}^3$ (i.e. the above linear combination of T, N and B is constant for all t). Show that the curve given by **r** lies always in the plane of T and N.

Hint: Differentiate the given relation and try to conclude that the torsion of the curve is always zero.

$k(t) \neq 0$ $\quad T'(t) = kN \quad N' = -kT + \tau B \quad B' = -\tau N$

$N(t) \cdot \left(2\cos(2t)T(t) + \sin(2t)T'(t) - 2\sin(2t)N(t) + \cos(2t)N'(t) + \underbrace{B'(t)}_{-\tau N}\right) = 0$

inner product with $N(t)$

$\Rightarrow \cancel{\sin(2t)\, T'(t) \cdot N(t)} - 2\sin(2t) + B'(t) \cdot N(t) = 0$ $\quad$ 6

$\Rightarrow \sin(2t)(k(t)N(t) \cdot N(t)) - 2\sin(2t) + -\tau = 0$ $\quad$?

$\Rightarrow \underbrace{\sin(2t)k(t) - 2\sin(2t)}_{0} = \tau$

$-\tau = \sin(2t)$

$N' \cdot B \quad -N \cdot B'$

$\Rightarrow \tau = 0$

$\boxed{\text{Since } \tau = 0\text{, the curve is always spanned on } T \text{ and } N \text{ plane}}$