THERE ARE FIVE QUESTIONS. EACH QUESTION IS WORTH TEN POINTS. YOUR SCORE FOR THE MIDTERM IS THE SUM OF ALL FIVE QUESTION SCORES.

SHOW ALL YOUR WORK.

ALL UNIVERSITY AND DEPARTMENTAL POLICIES REGARDING ACADEMIC INTEGRITY APPLY.

Question:	1	2	3	4	5	Total / 50
Score:	8	10	0	3	1	22

Question 1. (10 points)

Find a parametrization of the line determined by the intersection of planes Π_1, Π_2,

$(2, 1, 4)$

$$\Pi_1 : x - y + z = 5,$$

$2 - 1 + 4$

$$\Pi_2 : 2x - y + z = 0.$$

$4 - 1 + 4$

$n_1 = \langle 1, -1, 1 \rangle$

$n_2 = \langle 2, -1, 1 \rangle$

$$n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 1 & -1 & 1 \\ 2 & -1 & 1 \end{vmatrix} = i(-1+1) - j(1-2) + k(-1+2)$$

$\langle 0, 1, 1 \rangle$

$$\boxed{r(t) = \underline{\langle 2, 1, 4 \rangle} + t\langle 0, 1, 1 \rangle}$$

X

Question 2. (10 points)

Let $\mathcal{C}$ be determined by $y = (2/3)x^{3/2}$, $0 \le x \le 2$, that is,

$$\mathcal{C} = \left\{(x, y) \in \mathbb{R}^2 : y = \tfrac{2}{3}x^{\frac{3}{2}},\ x \in [0, 2]\right\}.$$

Find the arclength parametrization $\mathbf{r}_1(s)$ of $\mathcal{C}$ such that $\mathbf{r}_1(0) = \left\langle 0, \tfrac{2}{3}2^{\frac{3}{2}}\right\rangle$.

$u = t + 1$

$du = dt$

$r(t) = \langle t, \frac{2}{3}t^{\frac{3}{2}} \rangle \qquad r'(t) = \langle 1, t^{\frac{1}{2}} \rangle$

$\|r'(t)\| = \sqrt{1 + t} \qquad s = \int_0^t u^{\frac{1}{2}} \rightarrow \frac{2}{3}(t+1)^{\frac{3}{2}}\Big]_0^t$

$\frac{2}{3}(t+1)^{\frac{3}{2}} - \frac{2}{3}(1)^{\frac{3}{2}} = \frac{2}{3}\left[(t+1)^{\frac{3}{2}} - 1\right]$

$g^{-1}(s) \rightarrow t = \frac{2}{3}\left[(s+1)^{\frac{3}{2}} - 1\right] \qquad \frac{3}{2}t = (s+1)^{\frac{3}{2}} - 1$

$+1 \qquad +1$

$\frac{3}{2}t + 1 = (s+1)^{\frac{3}{2}} \qquad \left(\frac{3}{2}t + 1\right)^{\frac{2}{3}} = s + 1 \qquad g^{-1}(s) = \left(\frac{3}{2}s + 1\right)^{\frac{2}{3}} - 1$

$-1 \qquad -1$

$r_1(s) = \left\langle \left(\frac{3}{2}s + 1\right)^{\frac{2}{3}} - 1, \frac{2}{3}\left[\left(\frac{3}{2}s + 1\right)^{\frac{2}{3}} - 1\right]^{\frac{3}{2}} \right\rangle$

$r_1(0) = \left\langle (0+1) - 1, \frac{2}{3}\left[(1)^{\frac{2}{3}} - 1\right]^{\frac{3}{2}} \right\rangle$

$\boxed{r_1(0) = \langle 0, 0 \rangle}$?

Question 3. (10 points)

$(3x)^2 = 9x^2$

A particle moves in 3-space along a path parametrized by time as

$$\mathbf{r}(t) = \left\langle \sqrt{\sqrt{2}e^{-2t}+1}\cos(t), \sqrt{\sqrt{2}e^{-2t}+1}\sin(t), \frac{1}{\sqrt{2}}e^{-\frac{t^2}{2}} \right\rangle, \quad t \in [0,\infty).$$

When is the particle closest to the origin? where $\|r'(t)\| = 0$

$$r(t) = \langle (\sqrt{2}e^{-2}t+1)^{\frac{1}{2}}\cos t, (\sqrt{2}e^{-t}t+1)^{\frac{1}{2}}\sin t, \frac{1}{\sqrt{2}}e^{-\frac{t^2}{2}} \rangle$$

$$r'(t) = \langle \tfrac{1}{2}(\sqrt{2}e^{-2}t+1)^{-\frac{1}{2}} \cdot \sqrt{2}e^{-2} \cdot \cos t - (\sqrt{2}e^{-2}t+1)^{\frac{1}{2}}\sin t,$$
$$\tfrac{1}{2}(\sqrt{2}e^{-2}t+1)^{-\frac{1}{2}} \sqrt{2}e^{-2} \cdot \sin t + (\sqrt{2}e^{-2}t+1)^{\frac{1}{2}}\cos t,$$
$$\frac{1}{\sqrt{2}}e^{-\frac{t^2}{2}} \cdot -t \rangle$$

$$\|r'(t)\| = \sqrt{\tfrac{1}{2} \cdot (\sqrt{2}e^{-2}t+1)^{-1} \cdot 2e^{-4} \cdot \cos^2 t + (\sqrt{2}e^{-2}t+1)\sin^2 t + \tfrac{1}{2}(\sqrt{2}e^{-2}t+1)^{-1} \cdot 2e^{-4} \cdot \sin^2 t + (\sqrt{2}e^{-2}t+1)^{\frac{1}{2}}\cos^2 t + \tfrac{1}{2}e^{\frac{t^4}{4}} \cdot t^2}$$

Not what we want!

since domain $[0,\infty)$

$$\|r'(t)\|^2 = \frac{2e^{-4}}{2(\sqrt{2}e^{-2}t+1)} + (\sqrt{2}e^{-2}t+1) + \tfrac{1}{2}e^{\frac{t^4}{4}} \cdot t^2$$

$$0 = \frac{\cancel{2}e^{\cancel{-4}\,-2}}{\cancel{2}\sqrt{2}\cancel{e}t+\cancel{2}} + \frac{\sqrt{2}t}{e^2} + 1 + \tfrac{1}{2}e^{\frac{t^4}{4}} \cdot t^2$$

$$0 = \frac{1}{\sqrt{2}e^2t+2} + \frac{\sqrt{2}t}{e^2} + 1 + \frac{e^{\frac{t^4}{4}}t^2}{2}$$

Question 4. (10 points)

Suppose a vertical line $x = c$ is parametrized as

$$\mathbf{r}(t) = \langle c, y(t)\rangle, \quad t \in \mathbb{R},$$

with $y'(t) > 0$ for all t.

(a) Show that

$$\kappa(t) = \frac{\|\mathbf{T}'(t)\|}{v(t)} \equiv 0.$$

(b) Identify the arc length parametrization $\mathbf{r}_1(s)$ satisfying $\mathbf{r}_1(0) = \langle c, y(0)\rangle$, then show that

$$\kappa(s) = \|\mathbf{T}_1'(s)\| \equiv 0.$$

(Here $\mathbf{T}(t)$ is the unit tangent associated to $\mathbf{r}(t)$, $v(t)$ is speed associated to $\mathbf{r}(t)$, and $\mathbf{T}_1(s)$ is the unit tangent associated to $\mathbf{r}_1(s)$.)

a) $r'(t) = \langle \cancel{1}, y'(t) \rangle \quad T(t) = \dfrac{\langle 1, y'(t)\rangle}{\sqrt{1+(y'(t))^2}}$

$v(t) = \sqrt{y'(t)^2+1} \quad T(t) = \langle ((y(t))^2+1)^{-\frac{1}{2}}, y'(t)((y'(t))^2+1)^{-\frac{1}{2}} \rangle$

$T'(t) = \langle -\frac{1}{2}(y'(t)^2+1)^{-\frac{3}{2}} \cdot 2y'(t) \cdot y''(t),$

$y'(t)\left(-\frac{1}{2}(y'(t)^2+1)^{-\frac{3}{2}} \cdot 2y'(t) \cdot y''(t)\right) + y''(t)(y'(t)^2+1)^{-\frac{1}{2}} \rangle$

$$\frac{\|T'(t)\|}{v(t)} = \frac{\sqrt{\frac{1}{4}(y'(t)^2+1)^{-3} \cdot 4y'(t)^2 \cdot y''(t)^2 + y'(t)^2\left(-\frac{1}{4}(y'(t)^2+1)^{-3}\right) + y''(t)^2(y'(t)^2+1)^{-1}}}{\sqrt{y'(t)^2+1}}$$

$= 0$

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b) $r'(t) = \langle \cancel{1}, y'(t) \rangle \quad \|r'(t)\| = \sqrt{y(t)^2+1} \quad u = y'(t)^2+1$

$du = 2y'(t) \cdot y''(t)$

$s = \int_0^t \cancel{u^{\frac{1}{2}}} \to \frac{2}{3}(y'(t)^2+1)^{\frac{3}{2}} \Big]_0^t$

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Question 5. (10 points)

Parts (a) and (b) have nothing to do with each other.

(a) Define the average velocity of a parametrization of a curve $\mathbf{r}$ over interval $[0, t_1]$, $t_1 > 0$, as

$$\overline{\mathbf{v}} = \frac{1}{t_1} \int_0^{t_1} \mathbf{r}'(t)\, dt.$$

Write down an example of $\mathbf{r}$ and t_1 such that $\overline{\mathbf{v}} = \mathbf{0}$ but $\|\mathbf{r}'(t)\| > 0$ for all $t \in [0, t_1]$.

(b) For an arbitrary parametrization of a curve $\mathbf{r}$, show that $\mathbf{r}''(t) \cdot \mathbf{r}'(t) = v'(t)v(t)$.

) $\vec{r}(t) = \langle t, 1 \rangle \quad r'(t) = \langle 1, 0 \rangle \quad t_1 = 1$

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b) $r(t) = \langle a, b \rangle \quad r'(t) = \langle a', b' \rangle, \; r''(t) = \langle a'', b'' \rangle$

$\|r'(t)\| / v(t) = \sqrt{a'^2 + b'^2}, \; v'(t) = \frac{1}{2}(a'^2 + b'^2)^{-\frac{1}{2}} \cdot 2a'$

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b) $r(t) = \langle t, t \rangle \quad r'(t) = \langle 1, 1 \rangle, \quad r''(t) = \langle 0, 0 \rangle$

$v(t) = \sqrt{2} \qquad v'(t) = 0$

$$\boxed{\begin{array}{l} r'(t) \cdot r''(t) = 0 \\ v'(t)v(t) = 0. \end{array}}$$

So $r'(t) \cdot r''(t) = v'(t)v(t)$

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