THERE ARE FIVE QUESTIONS. EACH QUESTION IS WORTH TEN POINTS. YOUR SCORE FOR THE MIDTERM IS THE SUM OF ALL FIVE QUESTION SCORES.

SHOW ALL YOUR WORK.

ALL UNIVERSITY AND DEPARTMENTAL POLICIES REGARDING ACADEMIC INTEGRITY APPLY.

Question:	1	2	3	4	5	Total / 50
Score:	10	0	10	10	10	40

Question 1. (10 points)

0, 5,

(i) Equivalent vectors need not have the same direction. TRUE FALSE

(ii) If $P_1 = (2, -5, 2)$, $Q_1 = (2, 0, 1)$, and $Q_2 = (0, 5, -1)$, then $\overrightarrow{P_1Q_1}$ and $\overrightarrow{OQ_2}$ are equivalent. $\langle 0, 5, -1 \rangle$ TRUE FALSE

(iii) If two vectors have different terminal points, and the same initial point, then they can never be parallel. TRUE FALSE

(iv) If $\mathbf{u}, \mathbf{v}$ are non-zero, non-parallel vectors with the same base point, λ_1, λ_2 are scalars, and $\lambda_1\mathbf{u} + \lambda_2\mathbf{v} = \mathbf{0}$, then $\lambda_1 = \lambda_2 = 0$. TRUE FALSE

(v) Vectors $\mathbf{u} = \langle 1, 1, 0 \rangle$, $\mathbf{v} = \langle 0, -1, 0 \rangle$, and $\mathbf{w} = \langle 0, 0, 1 \rangle$, in that order, form a right-handed system. TRUE FALSE

$$\begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 0 & -1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -1 & 0 \end{vmatrix} i - \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} j + \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} k$$

$$i(0-0) - j(0+0) + (-1-0)k$$

$$\langle 0, 0, -1 \rangle$$

1 10

Question 2. (10 points)

$v \cdot w = \|v\| \|w\| \cos\theta$

Suppose $(\cos(\theta), \sin(\theta))$, $0 \le \theta \le \pi/2$, is a point on the unit circle in the first quadrant. Let α be such that $0 < \alpha < \pi/2$. Let $\mathbf{v} = \langle\cos(\theta), \sin(\theta)\rangle$, and $\mathbf{u} = \langle\cos(\theta+\alpha), \sin(\theta+\alpha)\rangle$.

(a) What is the (smaller) angle between $\mathbf{v}$ and $\mathbf{u}$?

(b) If $\mathbf{u}_{\|\mathbf{v}} = \lambda_1 \langle\cos(\theta), \sin(\theta)\rangle = \lambda_1 \mathbf{v}$, what is λ_1?

(c) If $\mathbf{u}_{\perp\mathbf{v}} = \lambda_2 \langle -\sin(\theta), \cos(\theta)\rangle$, what is λ_2?

(d) Write down the trignometric identities you recover from $\mathbf{u} = \mathbf{u}_{\|\mathbf{v}} + \mathbf{u}_{\perp\mathbf{v}}$.

a) Let's say $\theta = 0$ and $\alpha = \frac{\pi}{4}$; $\vec{v} = \langle 1, 0 \rangle$, $\vec{u} = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$

$\frac{\vec{v} \cdot \vec{u}}{\|\vec{v}\| \|\vec{u}\|} = \cos\theta$ angle between $\quad \frac{\frac{\sqrt{2}}{2} + 0}{1 \cdot 1} = \frac{\sqrt{2}}{2}$ $\quad \|v\| = \sqrt{1^2 + 0} = 1$

$\|u\| = \sqrt{\frac{2}{4} + \frac{2}{4}} = 1$

$\theta = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$ $\quad \boxed{\theta = \frac{\pi}{4}}$

b) Once again, $\theta = 0$ and $\alpha = \frac{\pi}{4}$: $u_{\|v} = \left(\frac{u \cdot v}{\|v\|^2}\right)\vec{v} = \left(\frac{\frac{\sqrt{2}}{2}}{1}\right)\langle 1, 0\rangle \rightarrow$

$\rightarrow \langle \frac{\sqrt{2}}{2}, 0 \rangle$ $\quad \boxed{\lambda_1 = \frac{\sqrt{2}}{2}}$ $\quad u_{\|v}$

c) with $\theta = \frac{\pi}{6}$ and $\alpha = \frac{\pi}{6}$: $u = u_{\|v} + u_{\perp v} \rightarrow u_{\perp v} = u - u_{\|v} \rightarrow$ $\quad \frac{\pi}{6} + \frac{\pi}{6} = \frac{2\pi}{6} = \frac{\pi}{3}$

$v = \langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$ $\quad u_{\|v} = \left(\frac{u \cdot v}{\|v\|^2}\right)\vec{v}$ $\quad u \cdot v = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$

$u = \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle$ $\quad \|v\|^2 = \frac{3}{4} + \frac{1}{4} = 1$

$u_{\|v} = \frac{\sqrt{3}}{2}\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle = \langle \frac{3}{4}, \frac{\sqrt{3}}{4} \rangle \rightarrow u_{\perp v} = \langle \frac{2}{4}, \frac{\sqrt{3}}{2} \rangle - \langle \frac{3}{4}, \frac{\sqrt{3}}{4} \rangle$

$u_{\perp v} = \langle -\frac{1}{4}, \frac{2\sqrt{3}}{4} - \frac{\sqrt{3}}{4} \rangle = \langle -\frac{1}{4}, \frac{\sqrt{3}}{4} \rangle$ $\quad \lambda_2 \langle -\frac{1}{2}, \frac{\sqrt{3}}{2} \rangle = \langle -\frac{1}{4}, \frac{\sqrt{3}}{4} \rangle$

$\boxed{\lambda_2 = \frac{1}{2}}$

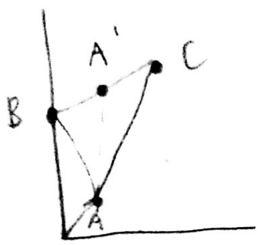
d)

Question 3. (10 points)

(a) Let $A = (1,1)$, $B = (0,3)$, $C = (3,4)$, and A' be the midpoint of the segment $\overline{BC}$. Write down a parametrization of the line through A and A' (in components).

(b) Let $\mathbf{u} = \overrightarrow{OB}$, $\mathbf{v} = \overrightarrow{OA}$, and $\mathbf{w} = \overrightarrow{OC}$. With reference to your paramertrization from part (a), at what parameter value does the line intersect the point ~~given by the head~~ of $\mathbf{v} + \mathbf{u} + \mathbf{w}$?

a)



$\vec{OA'} = \vec{OA} + \vec{AB} + \frac{1}{2}\vec{BC}$

$\vec{AA'} = A' - A$

$\left(\frac{0+3}{2}, \frac{3+4}{2}\right)$ $\quad A' = \left(\frac{3}{2}, \frac{7}{2}\right)$

$\vec{AA'} = \left(\frac{1}{2}, \frac{5}{2}\right)$

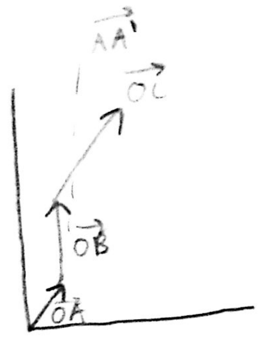
$\cancel{\vec{AA'} = \langle 1,1\rangle + t\langle}$

$\vec{AA'} = \langle \frac{3}{2}-1, \frac{7}{2}-1\rangle$

$\vec{AA'} = \langle \frac{1}{2}, \frac{5}{2}\rangle$

$\boxed{\vec{r}(t) = \langle 1,1\rangle + t\langle \frac{1}{2}, \frac{5}{2}\rangle}$

b)



$\langle 1,1\rangle + \langle 0,3\rangle + \langle 3,4\rangle = \langle 4,8\rangle$

$\langle 1,1\rangle + t\langle \frac{1}{2}, \frac{5}{2}\rangle = \langle 4,8\rangle$

$t\langle \frac{1}{2}, \frac{5}{2}\rangle = \langle 3,7\rangle$

$\vec{r}(t)$ does not intersect with the head of v+u+w

Question 4. (10 points)

Let $\mathbf{v} = \langle 0, 2, 1 \rangle$, and $\mathbf{w} = \langle 1, 0, 1 \rangle$. Write down a parametrization of the line $\mathcal{L}$ through the origin defined by the following requirement: if $P \in \mathcal{L}$, then $\overrightarrow{OP}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$.

Cross product

$$\vec{v} \times \vec{w} = \begin{vmatrix} i & j & k \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{vmatrix} =$$

$$i \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} - j \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} + k \begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} =$$

$$i(2-0) - j(0-1) + k(0-2) = 2i + j - 2k = \langle 2, 1, -2 \rangle$$

$$\boxed{\vec{r}(t) = \langle 0, 0, 0 \rangle + t\langle 2, 1, -2 \rangle}$$

✓

Question 5. (10 points)

Calculate the volume of the parallelepiped spanned by

$$\mathbf{u} = \langle 1, 2, 1 \rangle, \quad \mathbf{v} = \langle 0, 1, 3 \rangle, \quad \text{and} \quad \mathbf{w} = \langle 0, 0, -2 \rangle.$$

$$|\vec{u} \cdot (\vec{v} \times \vec{w})| = \det \begin{pmatrix} \vec{u} \\ \vec{v} \\ \vec{w} \end{pmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{vmatrix} =$$

$$1 \begin{vmatrix} 1 & 3 \\ 0 & -2 \end{vmatrix} - 2 \begin{vmatrix} 0 & 3 \\ 0 & -2 \end{vmatrix} + 1 \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$$

$$(-2-0) - 2(0-0) + (0-0) = -2$$

$$\boxed{2u^3}$$

$$\begin{vmatrix} i & j & k \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{vmatrix} = i \begin{vmatrix} 1 & 3 \\ 0 & -2 \end{vmatrix} - j \begin{vmatrix} 0 & 3 \\ 0 & -2 \end{vmatrix} + k \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$$

$$i(-2-0) - j(0-0) + k(0-0) = -2j = \langle -2, 0, 0 \rangle$$

$$\langle 1, 2, 1 \rangle \cdot \langle -2, 0, 0 \rangle = -2 + 0 + 0$$