Math 31B - Sorin Popa - Winter 2020 Final Exam
Grade: 160/160

1. (40 points) (*a*) Let $f(x) = \frac{x^3}{x^2+1}$. Calculate $(f^{-1})'(-1/2)$.

(*b*) Consider the function $g(x) = e^{2\ln x} + e^{(\ln x)^2}$, for $x > 0$. Find the equation of the tangent line to the graph of $g(x)$ at $x = 1$.

(*c*) Calculate the limit $\lim_{x\to\infty} (\frac{\cos x}{x} + x \sin \frac{1}{x})$.

(*d*) Calculated the surface area of revolution of the curve $y = 4x/3$ about the x-axis over the interval $[0, 3]$ and the arclength of the curve $y = x^{3/2}$ over the interval $[0, 2]$.

(a) Instead of explicitly calculating the inverse,
we can use the fact that $(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$.
So all we need to know is $f'(x)$ and $f^{-1}(-\frac{1}{2})$.
First, calculate $f'(x)$ using the Quotient Rule:

$$f'(x) = \frac{(x^2+1)(x^3)' - (x^3)(x^2+1)'}{(x^2+1)^2} = \frac{(x^2+1)(3x^2)-(x^3)(2x)}{(x^2+1)^2} = \frac{3x^4+3x^2-2x^4}{(x^2+1)^2} = \frac{x^4+3x^2}{(x^2+1)^2}$$

Next, calculate $f^{-1}(-1/2)$ by finding a value x such that $f(x) = -1/2$:
Try the value -1: $f(-1) = \frac{(-1)^3}{(-1)^2+1} = \frac{-1}{1+1} = -1/2$. So $f^{-1}(-1/2) = -1$.
Now solve for $(f^{-1})'(-\frac{1}{2})$:

$$(f^{-1})'(-\tfrac{1}{2}) = \frac{1}{f'(f^{-1}(-1/2))} = \frac{1}{f'(-1)} = \frac{1}{\frac{(-1)^4+3(-1)^2}{((-1)^2+1)^2}} = \frac{((-1)^2+1)^2}{(-1)^4+3(-1)^2} = \frac{(1+1)^2}{1+3(1)} = \frac{2^2}{4} = \frac{4}{4} = \boxed{1}$$

(b) To get the tangent line to a graph of a function,
use the linearization equation $L(x) = g'(a)(x-a) + g(a)$
We need to find $g'(a)$, so calculate $g'(x)$:

$$g'(x) = (e^{2\ln x})(2\ln x)' + (e^{(\ln x)^2})((\ln x)^2)'$$
$$= (e^{2\ln x})(\tfrac{2}{x}) + (e^{(\ln x)^2})(\tfrac{2\ln x}{x})$$
$$g'(1) = (e^{2\ln 1})(\tfrac{2}{1}) + \cancel{(e^{(\ln 1)^2})(\tfrac{2\ln 1}{1})} = 2e^0 = 2$$
$$g(1) = e^{2\ln(1)} + e^{(\ln 1)^2} = e^0 + e^0 = 1+1 = 2$$
$$L(x) = 2(x-1)+2 = 2x-2+2 = 2x$$

So the equation of the tangent line is $\boxed{L(x) = 2x}$

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(c) The limit can be evaluated as the sum of two separate limits:

$\lim\limits_{x\to\infty}\left(\frac{\cos x}{x}\right)+\lim\limits_{x\to\infty}\left(x\sin\left(\frac{1}{x}\right)\right)$

First, evaluate $\lim\limits_{x\to\infty}\left(\frac{\cos x}{x}\right)$ using the Squeeze Theorem.

Since $\cos(x)$ is bounded $\left(-\frac{1}{x}\le\frac{\cos x}{x}\le\frac{1}{x}\right)$,

then $\lim\limits_{x\to\infty}\left(-\frac{1}{x}\right)\le\lim\limits_{x\to\infty}\left(\frac{\cos x}{x}\right)\le\lim\limits_{x\to\infty}\left(\frac{1}{x}\right)$.

$\lim\limits_{x\to\infty}\left(-\frac{1}{x}\right)$ and $\lim\limits_{x\to\infty}\left(\frac{1}{x}\right)$ both evaluate to 0,

So $0\le\lim\limits_{x\to\infty}\left(\frac{\cos x}{x}\right)\le 0 \Rightarrow \lim\limits_{x\to\infty}\left(\frac{\cos x}{x}\right)=0$

Next, evaluate $\lim\limits_{x\to\infty}\left(x\sin\left(\frac{1}{x}\right)\right)$ using L'Hôpital's Rule.

The limit can be rewritten as $\lim\limits_{x\to\infty}\left(\frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}\right)$, which is of indeterminate form $0/0$.

Take the derivative of the numerator and denominator:

$\lim\limits_{x\to\infty}\left(\frac{(\sin(1/x))'}{(1/x)'}\right)=\lim\limits_{x\to\infty}\left(\frac{(1/x)'\cos(1/x)}{(1/x)'}\right)=\lim\limits_{x\to\infty}\left(\frac{\cancel{-\frac{1}{x^2}}\cos\left(\frac{1}{x}\right)}{\cancel{-\frac{1}{x^2}}}\right)=\lim\limits_{x\to\infty}\cos(1/x)=\cos(0)=1$

So $\lim\limits_{x\to\infty}\left(\frac{\cos x}{x}+x\sin\left(\frac{1}{x}\right)\right)=0+1=\boxed{1}$

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Question 1 continued...

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(d) First, calculate the surface area of revolution of $y = \frac{4x}{3}$ about the x-axis over $[0, 3]$

We know that the surface area S over $[a, b]$ is expressed as $S = 2\pi\int_a^b f(x)\sqrt{1+f'(x)^2}\,dx$

Solve for S: $S = 2\pi\int_0^3 \frac{4}{3}x\sqrt{1+(4/3)^2}\,dx = 2\pi\int_0^3 4/3\,x\sqrt{1+\frac{16}{9}}\,dx = 2\pi\int_0^3 \frac{4}{3}x\sqrt{\frac{25}{9}}\,dx$

$[a,b]=[0,3]$

$f'(x) = 4/3$

$\Rightarrow 2\pi\int_0^3 \frac{4}{3}\cdot\frac{5}{3}x\,dx = 2\pi\cdot\frac{4}{3}\cdot\frac{5}{3}\int_0^3 x\,dx = \frac{40\pi}{9}\left(\frac{1}{2}x^2\right)\Big|_0^3$

$\Rightarrow \frac{40\pi}{9}\left(\frac{1}{2}(3)^2\right) = \frac{40\pi}{\cancel{9}}\cdot\frac{\cancel{9}}{2} = \boxed{20\pi}$ is the surface area.

Next, calculate the arc length of the curve $y = x^{3/2}$ over $[0, 2]$.

We know that the arc length s over $[a, b]$ is expressed as $s = \int_a^b \sqrt{1+f'(x)^2}\,dx$

$f(x) = x^{3/2} \Rightarrow f'(x) = 3/2\,x^{1/2}$, $[a,b] = [0, 2]$.

Solve for s: $s = \int_0^2 \sqrt{1+\left(\frac{3}{2}x^{1/2}\right)^2}\,dx = \int_0^2 \sqrt{1+\frac{9}{4}x}\,dx$ Use u-sub: $u = 1+\frac{9}{4}x$, $du = \frac{9}{4}dx$

Change bounds! $\Rightarrow \frac{4}{9}\int_1^{11/2}\sqrt{u}\,du = \frac{4}{9}\left(\frac{2}{3}u^{3/2}\right)\Big|_1^{11/2} = \frac{4}{9}\left[\frac{2}{3}\left(\frac{11}{2}\right)^{3/2} - \frac{2}{3}(1)^{3/2}\right]$

$= \frac{4}{9}\left[\frac{\cancel{2}}{3}\frac{11\sqrt{11}}{\cancel{2}\sqrt{2}} - \frac{2}{3}\right] = \frac{4}{9}\left[\frac{11\sqrt{22}}{6} - \frac{2}{3}\right] = \boxed{\frac{22\sqrt{22}-8}{27}}$ is the arc length.

2. (40 points) (*a*) Show that the improper integral $\int_e^\infty x^{-2}\ln x\,dx$ is convergent and calculate its value.

(*b*) Prove that the series $\sum_{n=3}^{\infty} n^{-2}\ln n$ is convergent.

(*c*) Show that the series $\sum_{n=3}^{\infty}(-1)^n n^{-1}\ln n$ is not absolutely convergent but it is conditionally convergent.

(*d*) Show that the series $\sum_{n=1}^{\infty}\frac{1}{2n-\sin^2 n}$ is divergent but $\sum_{n=1}^{\infty}\frac{1}{n^2-\sin^2 n}$ is convergent.

(a) To calculate improper integrals, substitute the bound at infinity with a value R, and take the limit of the integral as $R\to\infty$.

$\lim_{R\to\infty}\int_e^R x^{-2}\ln x\,dx$ Integrate by Parts: $u=\ln x$ $dv=x^{-2}dx$ $du=\frac{1}{x}dx$ $v=-x^{-1}$ $\int u\,dv = uv-\int v\,du$ - Just deal with indefinite integrals for now

$$\int x^{-2}\ln x\,dx = \ln x(-x^{-1})-\int -x^{-1}\frac{1}{x}dx = -\frac{\ln x}{x}-\int -\frac{1}{x^2}dx$$

$$\Rightarrow -\frac{\ln x}{x}-\frac{1}{x} = -\frac{\ln(x)+1}{x}$$

$$\lim_{R\to\infty}\left(-\frac{\ln(x)+1}{x}\right)\Big|_e^R = \lim_{R\to\infty}\left(-\frac{\ln(R)+1}{R}\right)+\left(+\frac{\ln(e)+1}{e}\right)=\lim_{R\to\infty}\left(-\frac{\ln(R)+1}{R}\right)+\frac{2}{e}$$

Use L'Hôpital's Rule: $\lim_{R\to\infty}\left(-\frac{(\ln(R)+1)'}{R'}\right)=\lim_{R\to\infty}\left(-\frac{\frac{1}{R}}{1}\right)=\lim_{R\to\infty}\left(-\frac{1}{R}\right)=0$

So the improper integral $\int_e^\infty x^{-2}\ln x\,dx$ is $\boxed{\text{convergent}}$ and has a value of $\boxed{\frac{2}{e}}$

(b) The Integral Test states that an infinite series $\sum_{n=N}^{\infty} f(n)$ is convergent if the improper integral $\int_N^\infty f(x)\,dx$ is also convergent.

Since $\int_e^\infty x^{-2}\ln x\,dx$ is convergent (and $N=3$ is the first integer greater than e),

$\boxed{\sum_{n=3}^{\infty} n^{-2}\ln n \text{ is also convergent.}}$

Question 2 continues on the next page...

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(c) We can use the Alternating Series Test to prove conditional convergence of an alternating series, and check $\sum |a_n|$ for absolute convergence.

First, check if $\sum a_n$ converges absolutely by checking if $\sum |a_n|$ converges.

$a_n = (-1)^n n^{-1} \ln n \Rightarrow |a_n| = n^{-1} \ln n$ (since $(-1)^n$ is not always positive)

so we have $\sum_{n=3}^{\infty} \frac{\ln n}{n}$. To prove divergence, use the Integral Test.

Solve $\int_3^{\infty} \frac{\ln x}{x} dx$: $\lim_{R\to\infty} \int_3^R \frac{\ln x}{x} dx$ Use u-sub: $u = \frac{1}{x}$, $du = \ln x \, dx$ $\Rightarrow \lim_{R\to\infty} \int_3^R u \, du$

$$\Rightarrow \lim_{R\to\infty} \left(\frac{1}{2}u^2\right)\Big|_3^R = \lim_{R\to\infty} \frac{1}{2}(R)^2 - \frac{1}{2}(3)^2 = \infty$$

Since the improper integral diverges, the series $\sum |a_n|$ also diverges.

So $\sum a_n$ $\boxed{\text{does not converge absolutely}}$, but it still can converge conditionally.

Use the Alternating Series Test to check for conditional convergence:

Given an alternating series $\sum_{n=N}^{\infty} (-1)^{n-1} b_n$, the series converges if b_n is positive, b_n is decreasing, and $\lim_{n\to\infty} b_n = 0$. (Our series is $\sum_{n=3}^{\infty} (-1)^n \frac{\ln n}{n}$)

$b_n = \frac{\ln n}{n}$ is always positive since $\ln n > 0$ and $n > 0$ for $n \geq 3$.

To check if b_n is decreasing for $n \geq 3$, take the derivative:

$$\left(\frac{\ln x}{x}\right)' = \frac{x(\ln x)' - (\ln x)(x')}{x^2} = \frac{x(\frac{1}{x}) - \ln x(1)}{x^2} = \frac{1 - \ln x}{x^2}$$

By inspection, $1 - \ln x > 0$ when $\ln x < 1 \Rightarrow x < e$, and $1 - \ln x < 0$ when $\ln x > 1 \Rightarrow x > e$.

This means that the derivative of $b_n < 0$ for $n \geq 3$, meaning b_n is decreasing.

Finally, use L'Hôpital's Rule to check if $\lim_{n\to\infty} b_n = 0$:

$$\lim_{x\to\infty} \left(\frac{(\ln x)'}{(x)'}\right) = \lim_{x\to\infty} \left(\frac{\frac{1}{x}}{1}\right) = \lim_{x\to\infty} \left(\frac{1}{x}\right) = 0$$

Since all criteria have been satisfied, the series is $\boxed{\text{conditionally convergent.}}$

Question 2 continued...

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(d) We can use Direct/Limit Comparison Tests to compare the two series to similar series that can be directly evaluated.

First, compare $\sum_{n=1}^{\infty} \frac{1}{2n-\sin^2 n}$ to $\sum_{n=1}^{\infty} \frac{1}{2n}$.

We know $\sum_{n=1}^{\infty} \frac{1}{2n}$ diverges by the p-test, and $\frac{1}{2n-\sin^2 n} > \frac{1}{2n}$ for all $n > 0$, since $2n - \sin^2 n < 2n$, because $\sin^2 n > 0$ for all $n \in \mathbb{R}$

By the Direct Comparison Test, if a smaller series diverges, then the larger one must diverge. Therefore, $\sum_{n=1}^{\infty} \frac{1}{2n-\sin^2 n}$ $\boxed{\text{diverges}}$.

Next, compare $\sum_{n=1}^{\infty} \frac{1}{n^2-\sin^2 n}$ to $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

We know $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-test, but we can't use DCT since it can only prove a series converges if a larger series converges, and $\frac{1}{n^2-\sin^2 n} > \frac{1}{n^2}$ for all $n > 0$.

Instead, use Limit Comparison Test:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n^2-\sin^2 n}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2-\sin^2 n} = \lim_{n\to\infty} \frac{1}{1-\left(\frac{\sin^2 n}{n^2}\right)}$$ Evaluate this part of the limit

$\lim_{n\to\infty} \frac{\sin^2 n}{n^2}$ is bounded by $\frac{0}{n^2} \le \frac{\sin^2 n}{n^2} \le \frac{1}{n^2}$ since $\sin^2 n$ is bounded $(0 \le \sin^2 n \le 1)$

$\lim_{n\to\infty} \frac{0}{n^2}$ and $\lim_{n\to\infty} \frac{1}{n^2}$ both evaluate to 0, so $\lim_{n\to\infty} \frac{\sin^2 n}{n^2} = 0$.

So $\lim_{n\to\infty} \frac{1}{1-\frac{\sin^2 n}{n^2}} = \frac{1}{1-0} = 1 = \left(L = \lim_{n\to\infty} \frac{a_n}{b_n}\right)$

By the LCT, if $L > 0$, then $\sum a_n$ converges if and only if $\sum b_n$ converges.

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, $\sum_{n=1}^{\infty} \frac{1}{n^2-\sin^2 n}$ $\boxed{\text{converges.}}$

3. (40 points) (*a*) Show that the improper integral $\int_2^3 (x-2)^{-1/3}dx$ is convergent and calculate its value.

(*b*) Calculate the indefinite integral $\int \frac{x^3+x}{x^2-x}dx$.

(*c*) Show that the series $\sum_{n=1}^{\infty} \frac{\sin(n^{-1/2})}{n}$ is convergent but $\sum_{n=1}^{\infty} \frac{\sin(n^{-1/2})}{n^{1/2}}$ is divergent.

(*d*) Calculate $\lim_{n\to\infty}(1+n^{-1})^{-n}$ and use this to calculate the radius of convergence of the series $\sum_{n=1}^{\infty}(1+n^{-1})^{-n}x^n$.

(a) The improper integral has a vertical asymptote at $x=2$. We can substitute the bound at $x=2$ with R, and find the limit of the integral as $R\to 2$.

$\lim_{R\to 2}\int_R^3 (x-2)^{-1/3}dx$ u-sub: $u=x-2$, $du=dx$ $\Rightarrow \lim_{R\to 2}\int_{R-2}^{1} u^{-1/3}du = \lim_{R\to 2}\left(3/2\, u^{2/3}\right)\Big|_{R-2}^{1}$

$\Rightarrow \frac{3}{2}(1)^{2/3} - \lim_{R\to 2}\left(3/2(R-2)^{2/3}\right) = \frac{3}{2} - \cancel{\frac{3}{2}(2-2)^{2/3}} = 3/2$

The improper integral is $\boxed{\text{convergent}}$ and has a value of $\boxed{\frac{3}{2}}$

(b) $\int \frac{x^3+x}{x^2-x}dx$ simplifies to: $\int \frac{x^2+1}{x-1}dx$ once you factor out an x.

Use u-sub: $u=x-1$, $du=dx$ $\Rightarrow \int \frac{(u+1)^2+1}{u}du = \int \frac{u^2+2u+2}{u}du$

Split the integral: $\Rightarrow \int \frac{u^2}{u}du + \int \frac{2u}{u}du + \int \frac{2}{u}du = \int u\,du + \int 2\,du + \int \frac{2}{u}du$

Evaluate each integral: $\Rightarrow \frac{1}{2}u^2 + 2u + 2\ln|u|$

Plug x back in: $\Rightarrow \frac{1}{2}(x-1)^2 + 2(x-1) + 2\ln|x-1| + C$

Simplify: $\frac{1}{2}(x^2-2x+1) + 2(x-1) + 2\ln|x-1| + C$

$\frac{1}{2}x^2 - x + \frac{1}{2} + 2x - 2 + 2\ln|x-1| + C$

$\frac{1}{2}x^2 + x + 2\ln|x-1| + \underbrace{\frac{1}{2} - 2 + C}_{'C}$

$= \boxed{\frac{1}{2}x^2 + x + 2\ln|x-1| + C}$

Question 3 continues on the next page...

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(c) Since $\sin\left(\frac{1}{\sqrt{n}}\right)$ is bounded, we can use the Direct Comparison Test to prove convergence/divergence of series containing $\sin\left(\frac{1}{\sqrt{n}}\right)$.

For all $n \geq 1$, $\sin\left(\frac{1}{\sqrt{n}}\right) < \frac{1}{\sqrt{n}}$ (since $\sin x < x$ for all positive x)

Therefore, $\frac{\sin\left(\frac{1}{\sqrt{n}}\right)}{n} < \frac{\frac{1}{\sqrt{n}}}{n}$. $\frac{\frac{1}{\sqrt{n}}}{n} = \frac{1}{n^{3/2}}$, $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is convergent by the p-test.

Since $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges and is bigger than $\frac{\sin\left(\frac{1}{\sqrt{n}}\right)}{n}$, $\boxed{\sum_{n=1}^{\infty} \frac{\sin(n^{-1/2})}{n} \text{ converges}}$

For all $n \geq 1$, $\frac{1}{2\sqrt{n}} < \sin\left(\frac{1}{\sqrt{n}}\right)$.

Rationale: $\sin\left(\frac{1}{\sqrt{x}}\right)$ approaches $\frac{1}{\sqrt{x}}$ but is always greater than $\frac{1}{\sqrt{x}}$. $\sin\left(\frac{1}{2\sqrt{x}}\right)$ approaches $\frac{1}{\sqrt{x}}$ but is always less than $\frac{1}{\sqrt{x}}$ for $x \geq 1$.

Therefore, $\frac{\frac{1}{2\sqrt{n}}}{\sqrt{n}} < \frac{\sin\left(\frac{1}{\sqrt{n}}\right)}{\sqrt{n}}$.

$\frac{\frac{1}{2\sqrt{n}}}{\sqrt{n}} = \frac{1}{2n}$, $\sum_{n=1}^{\infty} \frac{1}{2n}$ diverges by the p-test.

Since $\sum_{n=1}^{\infty} \frac{1}{2n}$ diverges and is smaller than $\frac{\sin\left(\frac{1}{\sqrt{n}}\right)}{\sqrt{n}}$, $\boxed{\sum_{n=1}^{\infty} \frac{\sin(n^{-1/2})}{n^{1/2}} \text{ diverges.}}$

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(d) We can manipulate $\lim_{n\to\infty}(1+n^{-1})^{-n}$ using the change-of-base formula.

$(1+n^{-1})^{-n} = e^{\ln(1+n^{-1})^{-n}} = e^{-n\ln(1+n^{-1})}$. Now we can just look at $\lim_{n\to\infty} -n\ln(1+n^{-1})$.

$\lim_{n\to\infty} -n\ln(1+n^{-1}) = \lim_{n\to\infty} \frac{\ln(1+n^{-1})}{-n^{-1}} \xrightarrow{\text{L'Hôpital}} \lim_{n\to\infty} \frac{\cancel{-n^{-2}}\left(\frac{1}{1+n^{-1}}\right)}{\cancel{n^{-2}}} = \lim_{n\to\infty} -\frac{1}{1+\frac{1}{n}} = -1$

So $\lim_{n\to\infty} e^{\ln(1+n^{-1})^{-n}} = e^{-1} = \boxed{\frac{1}{e}}$. When $n\to\infty$, $(1+n^{-1})^{-n}x^n = \frac{x^n}{e}$

To calculate the radius of convergence, use the Ratio Test:

$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{\cancel{e}}\frac{\cancel{e}}{x^n}\right| = \left|\frac{\cancel{x^n}\cdot x}{\cancel{x^n}}\right| = |x|$ $\rho < 1$ when $|x| < 1$, so: The radius of convergence $\boxed{R = 1}$

4. (40 points) (*a*) Calculate the series $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$.

(*b*) Calculate the series $\sum_{n=1}^{\infty} \frac{4-3\cdot 2^n}{5^{n-1}}$

(*c*) Calculate the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{2^n}{n!} x^{2n}$.

(*d*) Find the Taylor polynomial T_n of degree n for $f(x) = \ln x$ with center at $a = 1$. Use this and the Error Bound Theorem to show that $|\ln 1.1 - T_3(1.1)| < 10^{-4}$.

(a) $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ is a telescoping series, which means if we analyze a partial sum from that series, we can cancel terms out so that we always get a finite number of terms.

To make the terms cancel, we rewrite $\frac{1}{n^2+n}$ using Partial Fraction Decomposition.

$\frac{1}{n^2+n} \Rightarrow \frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} \Rightarrow A(n+1) + B(n) = 1$ (after mult. by $n(n+1)$)

Set $n=0$: $A(0+1) + \cancel{B(0)} = 1 \Rightarrow A = 1$

Set $n=-1$: $\cancel{A(-1+1)} + B(-1) = 1 \Rightarrow B = -1$

So $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$. Let's look at a partial sum with N terms:

$$\sum_{n=1}^{N} \frac{1}{n} - \frac{1}{n+1} = \underbrace{\frac{1}{1} - \cancel{\frac{1}{2}}}_{n=1} + \underbrace{\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}}_{n=2} + \underbrace{\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}}_{n=3} + \dots + \underbrace{\cancel{\frac{1}{N-1}} - \cancel{\frac{1}{N}}}_{n=N-1} + \underbrace{\cancel{\frac{1}{N}} - \frac{1}{N+1}}_{n=N}$$

All terms except for $\frac{1}{1}$ and $-\frac{1}{N+1}$ cancel out, no matter what N is!

So the partial sum for N terms is always $1 - \frac{1}{N+1}$.

To find $\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$, just find the partial sum of an infinite number of terms (which is an infinite sum.)

Analyze the partial sum as $N \to \infty$:

$$\lim_{N\to\infty} \left(1 - \frac{1}{N+1}\right) = 1 - 0 = \boxed{1}$$

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(b) $\sum_{n=1}^{\infty} \frac{4-3(2^n)}{5^{n-1}}$ can be broken down into two easy-to-evaluate geometric series:

$\Rightarrow \sum_{n=0}^{\infty} \frac{4-3(2^{n+1})}{5^n} = \sum_{n=0}^{\infty} \frac{4}{5^n} - \sum_{n=0}^{\infty} \frac{3(2^n)(2^1)}{5^n} = \sum_{n=0}^{\infty} 4 \cdot \frac{1}{5^n} - \sum_{n=0}^{\infty} 6 \cdot \frac{2^n}{5^n} = \sum_{n=0}^{\infty} 4 \cdot \left(\frac{1}{5}\right)^n - \sum_{n=0}^{\infty} 6 \cdot \left(\frac{2}{5}\right)^n$

We know that the sum of a geometric series is $\sum_{n=0}^{\infty} cr^n = \frac{c}{1-r}$ where $|r| < 1$.

$\sum_{n=0}^{\infty} 4\left(\frac{1}{5}\right)^n - \sum_{n=0}^{\infty} 6\left(\frac{2}{5}\right)^n = \frac{4}{1-\frac{1}{5}} - \frac{6}{1-\frac{2}{5}} = \frac{4}{\frac{4}{5}} - \frac{6}{\frac{3}{5}} = \cancel{4} \cdot \frac{5}{\cancel{4}} - 6 \cdot \frac{5}{3} = 5 - 10 = \boxed{-5}$

(c) We can use the Ratio Test to find the radius of convergence of a power series.

Let $a_n = \frac{2^n}{n!} x^{2n}$. $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a}\right| = \lim_{n\to\infty} \left| \frac{2^{n+1} x^{2n+2}}{(n+1)!} \cdot \frac{n!}{2^n x^{2n}} \right| = \lim_{n\to\infty} \left| \frac{\cancel{2^n} \cdot 2 \cdot \cancel{x^{2n}} \cdot x^2}{(n+1)\cancel{n!}} \cdot \frac{\cancel{n!}}{\cancel{2^n}\cancel{x^{2n}}} \right|$

$\Rightarrow \lim_{n\to\infty} \left|\frac{2x^2}{n+1}\right| = 0$ for all x.

Since $\rho = 0$ for all x, the series converges for all x, meaning the radius of convergence $\boxed{R = \infty}$

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(d) The Taylor polynomial T_n of degree n for a function $f(x)$ with center a is

$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$. $f(x) = \ln x$ and the center is at $a = 1$

$f(x) = \ln x$ $\quad f'(x) = x^{-1}$ $\quad f''(x) = -x^{-2}$ $\quad f^{(3)}(x) = 2x^{-3}$ $\quad f^{(4)}(x) = -6x^{-4}$

$f(1) = \ln(1) = 0$ $\quad f'(1) = 1$ $\quad f''(1) = -1$ $\quad f^{(3)}(1) = 2$ $\quad f^{(4)}(x) = -6$

So $\ln(x) = 0 + 1(x-1) - \frac{1(x-1)^2}{2!} + \frac{2(x-1)^3}{3!} - \frac{6(x-1)^4}{4!} \ldots$

By observing the pattern, we can see that $f^{(k)}(a) = (-1)^{k-1}(k-1)!$ for $k \geq 1$

So $T_n(x) = \sum_{k=1}^{n} \frac{(-1)^{k-1}(k-1)!}{\underbrace{k!}_{k(k-1)!}}(x-1)^k = \boxed{\sum_{k=1}^{n} \frac{(-1)^{k-1}(x-1)^k}{k}}$

The Error Bound Theorem states that

$|f(x) - T_n(x)| \leq K \frac{|x-a|^{n+1}}{(n+1)!}$ where $|f^{(n+1)}(u)| \leq K$ for all u between a and x.

<u>Find error bound:</u> $n=3$ $\quad |\ln 1.1 - T_3(1.1)| \leq K \frac{|1.1-1|^4}{4!} \Rightarrow K \frac{|0.1|^4}{4!} \Rightarrow \frac{K \cdot 10^{-4}}{4!}$

Find the maximum value of $|f^4(u)|$ where u is in the interval $[1, 1.1]$:

$f^{(4)}(x) = -6x^{-4}$, $f^{(5)}(x) = 24x^5 \Rightarrow$ on $[1, 1.1]$ $f^{(4)}(x)$ is increasing.

Since $f^{(4)}(x)$ is negative on $[1, 1.1]$, $|f^{(4)}(x)|$ is decreasing for all u.

The maximum value of $|f^{(4)}(x)|$ is therefore at $x=1$, and is $|-6(1)^{-4}| = 6$.

So $K = 6$, and the error bound $\boxed{|\ln 1.1 - T_3(1.1)| \leq \frac{6 \cdot 10^{-4}}{4!} = \frac{10^{-4}}{4} < 10^{-4}}$