Math 31B
Integration and Infinite Series

Midterm 1

Instructions: You have 50 minutes to complete this exam. There are four questions, worth a total of 50 points. This test is closed book and closed notes. No calculator is allowed.

For full credit show all of your work legibly. Please write your solutions in the space below the questions; INDICATE if you go over the page and/or use scrap paper.

Do not forget to write your name, discussion and UID in the space below.

Name: ______________________________

Student ID number: ________________________

Discussion: ____________________________

Question	Points	Score
1	10	
2	10	
3	14	
4	16	
Total:	50	

Problem 1.

Calculate the following integrals (the first is definite, the second is indefinite).

(a) [4pts.] $\int_1^{\sqrt{8}} \ln(e^{2x})\ dx$.

Solution:

$$\int_1^{\sqrt{8}} \ln(e^{2x})\ dx = \int_1^{\sqrt{8}} 2x\ dx = \Big[x^2\Big]_1^{\sqrt{8}} = 8 - 1 = 7.$$

(b) [6pts.] $\int \frac{\ln(\ln x)}{x \ln x}\ dx$.

Solution: Let $u = \ln(\ln x)$. Then $du = \frac{1}{x \ln x}\ dx$. So

$$\int \frac{\ln(\ln x)}{x \ln x}\ dx = \int u\ du = \frac{u^2}{2} + c = \frac{(\ln(\ln x))^2}{2} + c.$$

Problem 2.

Calculate the following limits using any techniques you like.

(a) [4pts.] $\lim_{x\to\infty}(1 + (\frac{2}{3})^x)^{\frac{1}{x}}$.

Solution:

$$\lim_{x\to\infty}\left(1 + \left(\frac{2}{3}\right)^x\right)^{\frac{1}{x}} = \left(1 + \lim_{x\to\infty}\left(\frac{2}{3}\right)^x\right)^{\lim_{x\to\infty}\frac{1}{x}} = (1+0)^0 = 1.$$

(b) [6pts.] $\lim_{x\to 0+}(1 + 2x)^{\frac{1}{x}}$.

Solution: Let $y(x) = \ln(1+2x)^{\frac{1}{x}} = \frac{\ln(1+2x)}{x}$. Then

$$\lim_{x\to 0+} y(x) = \lim_{x\to 0+} \frac{\ln(1+2x)}{x} = \lim_{x\to 0+} \frac{\frac{2}{1+2x}}{1} = 2.$$

So $\lim_{x\to 0+}(1+2x)^{\frac{1}{x}} = e^{\lim_{x\to 0+} y(x)} = e^2$.

Problem 3.

Throughout this question you can use any familiar results from lectures as long as you explain why they are applicable.

(a) [3pts.] What is the sequence of partial sums for $\sum_{n=1}^{\infty} 1$?
Does this series converge or diverge? Explain. If so, what to?

Solution: The N-th term of the sequence of partial sums is given by $s_N = N$. Since $\lim_{N\to\infty} s_N = \lim_{N\to\infty} N = \infty$, the sequence of partial sums diverges. This is precisely what we mean by saying the series $\sum_{n=1}^{\infty} 1$ diverges.

(b) [3pts.] The partial sums of a series $\sum_{n=1}^{\infty} a_n$ are given by

$$s_N = \frac{1}{N}.$$

Does the series converge? Explain. If so, what to?

Solution: Since $\lim_{N\to\infty} s_N = \lim_{N\to\infty} \frac{1}{N} = 0$, the sequence of partial sums converges to 0. This is precisely what we mean by saying the series $\sum_{n=1}^{\infty} a_n$ converges to 0.

(c) [3pts.] The partial sums of another series $\sum_{n=1}^{\infty} a_n$ are given by

$$s_N = \sum_{n=1}^{N} \frac{1}{n}.$$

Does the series converge? Explain. If so, what to?

Solution: We see that $a_n = \frac{1}{n}$, so $\sum_{n=1}^{\infty} a_n$ is the harmonic series, and this is known to diverge.

(d) [5pts.] Consider the series $\sum_{n=1}^{\infty}(\frac{1}{n} - \frac{1}{n+2})$.
Calculate the partial sums s_4 and s_6. Does the series converge? If so, what to?

Solution:

$$s_4 = \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) = \frac{3}{2} - \frac{1}{5} - \frac{1}{6}.$$

$$s_6 = \left(\frac{3}{2} - \frac{1}{5} - \frac{1}{6}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \left(\frac{1}{6} - \frac{1}{8}\right) = \frac{3}{2} - \frac{1}{7} - \frac{1}{8}.$$

In general, $s_N = \frac{3}{2} - \frac{1}{N+1} - \frac{1}{N+2}$. The series $\sum_{n=1}^{\infty}(\frac{1}{n} - \frac{1}{n+2})$ converges to

$$\lim_{N\to\infty} s_N = \lim_{N\to\infty} \left(\frac{3}{2} - \frac{1}{N+1} - \frac{1}{N+2}\right) = \frac{3}{2}.$$

Problem 4.

Prove the convergence or divergence of the following series using any rules or tests that work. You should verify the hypotheses of any test that you use carefully. In particular, inequalities and limit calculations should be justified.

(a) [4pts.] $\sum_{n=1}^{\infty} \frac{1}{n5^n}$

Solution: Let $\text{SMALL}_n = \frac{1}{n5^n}$ and $\text{BIG}_n = \frac{1}{5^n}$.
$\sum_{n=1}^{\infty} \text{BIG}_n$ converges since it is a geometric series with $r = \frac{1}{5} < 1$.
We have $0 \leq \text{SMALL}_n \leq \text{BIG}_n$ because $n \geq 1$.
The direct comparison test tells us that $\sum_{n=1}^{\infty} \text{SMALL}_n = \sum_{n=1}^{\infty} \frac{1}{n5^n}$ converges.

(b) [4pts.] $\sum_{n=1}^{\infty} \frac{4^n+n^2}{2^n-n}$

Solution: Let $\text{SMALL}_n = \frac{4^n}{2^n}$ and $\text{BIG}_n = \frac{4^n+n^2}{2^n-n}$.
$\sum_{n=1}^{\infty} \text{SMALL}_n$ diverges since it is a geometric series with $r = 2 \geq 1$.
Since BIG_n has a larger numerator and smaller denominator than SMALL_n, we have $0 \leq \text{SMALL}_n \leq \text{BIG}_n$
The direct comparison test tells us that $\sum_{n=1}^{\infty} \text{BIG}_n = \sum_{n=1}^{\infty} \frac{4^n+n^2}{2^n-n}$ diverges.

(c) [4pts.] $\sum_{n=1}^{\infty} \frac{n^3}{\sqrt{n^7+3n^2}}$.

Solution: Let $a_n = \frac{n^3}{\sqrt{n^7+3n^2}}$ and $b_n = \frac{1}{\sqrt{n}}$. Then $a_n, b_n > 0$ for all n, and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^3\sqrt{n}}{\sqrt{n^7+3n^2}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{3}{n^5}}} = 1 \neq 0, \infty.$$

$\sum_{n=1}^{\infty} b_n$ diverges by the p-test, since $p = \frac{1}{2} \leq 1$.
The limit comparison tests says that $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n^3}{\sqrt{n^7+3n^2}}$ diverges.

(d) [4pts.] $\sum_{n=1}^{\infty} \frac{1}{100(\ln(n)+\sqrt{n})}$.

Solution: Let $a_n = \frac{1}{100(\ln(n)+\sqrt{n})}$ and $b_n = \frac{1}{\sqrt{n}}$. Then $a_n, b_n > 0$ for all n, and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n}}{100(\ln(n)+\sqrt{n})} = \lim_{n\to\infty} \frac{1}{100(\frac{\ln(n)}{\sqrt{n}}+1)} = \frac{1}{100} \neq 0, \infty.$$

($\lim_{n\to\infty} \frac{\ln(n)}{\sqrt{n}} = 0$ is obtained by one application of L'Hôpital's rule.)
$\sum_{n=1}^{\infty} b_n$ diverges by the p-test, since $p = \frac{1}{2} \leq 1$.
The limit comparison tests says that $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{100(\ln(n)+\sqrt{n})}$ diverges.