Math 31B
Integration and Infinite Series

Midterm 1

Instructions: You have 50 minutes to complete this exam. There are four questions, worth a total of 50 points. This test is closed book and closed notes. No calculator is allowed.

For full credit show all of your work legibly. Please write your solutions in the space below the questions; INDICATE if you go over the page and/or use scrap paper.

Do not forget to write your name, discussion and UID in the space below.

Name:
Student ID
Discussion

Question	Points	Score
1	10	10
2	10	7
3	14	14
4	16	13
Total:	50	44

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Problem 1.

Calculate the following integrals (the first is definite, the second is indefinite).

(a) [4pts.] $\int_1^{\sqrt{8}} \ln(e^{2x})\, dx$.

(b) [6pts.] $\int \frac{\ln(\ln x)}{x \ln x}\, dx$.

a. $\int_1^{\sqrt{8}} 2x \ln e\, dx = \int_1^{\sqrt{8}} 2x\, dx$

$= x^2 \Big]_1^{\sqrt{8}} = (\sqrt{8})^2 - 1^2 = 8 - 1 = \boxed{7}$

b. $u = \ln x$

$du = \frac{1}{x} dx$

$\int \frac{\ln u}{u} du$ $\quad v = \ln u$

$dv = \frac{1}{u} dv$

$= \int v\, dv = \frac{1}{2} v^2 = \frac{1}{2}(\ln u)^2 = \boxed{\frac{1}{2}(\ln(\ln x))^2 + C}$

Problem 2.

Calculate the following limits using any techniques you like.

1 (a) [4pts.] $\lim_{x\to\infty}(1+(\frac{2}{3})^x)^{\frac{1}{x}}$.

6 (b) [6pts.] $\lim_{x\to 0+}(1+2x)^{\frac{1}{x}}$.

$$y = \lim_{x\to\infty}\left(1+\left(\tfrac{2}{3}\right)^x\right)^{1/x}$$

a. $$\ln y = \ln\left(1+\left(\tfrac{2}{3}\right)^x\right)\left(\tfrac{1}{x}\right)$$

$$= \frac{\ln\left(1+\left(\frac{2}{3}\right)^x\right)}{x}$$

$\cancel{\lim_{x\to\infty}}$ $\ln\left(1+\left(\frac{2}{3}\right)^x\right)$ can never be equal to $\ln(1)$ because $\left(\frac{2}{3}\right)^\infty$ will always be a value greater than zero. Also, since ln grows slowly, $\lim_{x\to\infty}\frac{1}{x}$ overpowers it, so $\boxed{\lim_{x\to\infty}\left(1+\left(\frac{2}{3}\right)^x\right)^{\frac{1}{x}}=0}$

not really...

b. $$y = \lim_{x\to 0^+}(1+2x)^{\frac{1}{x}}$$

$$\ln y = \frac{1}{x}\ln(1+2x)$$

$$= \frac{\ln(1+2x)}{x}$$

$$\lim_{x\to 0^+}\frac{\ln(1+2x)}{x} = \frac{\ln(1)}{0} = \frac{0}{0} \quad \text{L'Hopital}$$

$$\ln y' = \frac{\frac{1}{1+2x}(2)}{1} = \frac{2}{1+2x}$$

$$\lim_{x\to 0^+}\frac{2}{1+2x} = \frac{2}{1+0} = 2$$

$$\ln y = 2$$

$$e^{\ln y} = e^2$$

$$\lim_{x\to 0^+}(1+2x)^{\frac{1}{x}} = \boxed{e^2}$$

Problem 3.

Throughout this question you can use any familiar results from lectures as long as you explain why they are applicable.

(a) [3pts.] What is the sequence of partial sums for $\sum_{n=1}^{\infty} 1$?
Does this series converge or diverge? Explain. If so, what to?

$s_1 = 1$
$s_2 = 1+1 = 2$
$s_3 = 1+1+1 = 3$

$s_N = n$

$\boxed{\lim_{n\to\infty} s_N = \lim_{n\to\infty} n = \lim_{x\to\infty} x \text{ diverges to } \infty, \text{ therefore, } \sum_{n=1}^{\infty} 1 \text{ diverges to } \infty \text{ as well}}$

(b) [3pts.] The partial sums of a series $\sum_{n=1}^{\infty} a_n$ are given by

$$s_N = \frac{1}{N}.$$

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Does the series converge? Explain. If so, what to?

$\lim_{n\to\infty} \frac{1}{N} = \lim_{x\to\infty} \frac{1}{x}$ converges to 0,
therefore $\sum_{n=1}^{\infty} a_n$ converges to 0 as well

(c) [3pts.] The partial sums of another series $\sum_{n=1}^{\infty} a_n$ are given by

$$s_N = \sum_{n=1}^{N} \frac{1}{n}.$$

Does the series converge? Explain. If so, what to?

s_N diverges because $\sum_{n=1}^{N} \frac{1}{n}$ is the harmonic series,
therefore $\sum_{n=1}^{\infty} a_n$ diverges, so the series does
not converge.

(d) [5pts.] Consider the series $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+2}\right)$.
Calculate the partial sums s_4 and s_6.
Does the series converge? If so, what to?

$$S_1 = 1 - \frac{1}{3} = \frac{2}{3}$$

$$S_2 = \frac{2}{3} + \left(\frac{1}{2} - \frac{1}{4}\right) = \frac{2}{3} + \frac{1}{4} = \frac{8}{12} + \frac{3}{12} = \frac{11}{12}$$

$$S_4 = \left(1 - \cancel{\tfrac{1}{3}}\right) + \left(\frac{1}{2} - \cancel{\tfrac{1}{4}}\right) + \left(\cancel{\tfrac{1}{3}} - \frac{1}{5}\right) + \left(\cancel{\tfrac{1}{4}} - \frac{1}{6}\right)$$

$$= 1 + \frac{1}{2} - \frac{1}{5} - \frac{1}{6} = \frac{3}{2} - \frac{6}{30} - \frac{5}{30} = \frac{45}{30} - \frac{11}{30} = \boxed{\frac{34}{30}}$$

$$S_6 = \left(1 - \cancel{\tfrac{1}{3}}\right) + \left(\frac{1}{2} - \cancel{\tfrac{1}{4}}\right) + \left(\cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{5}}\right) + \left(\cancel{\tfrac{1}{4}} - \cancel{\tfrac{1}{6}}\right) + \left(\cancel{\tfrac{1}{5}} - \frac{1}{7}\right) + \left(\cancel{\tfrac{1}{6}} - \frac{1}{8}\right)$$

$$= 1 + \frac{1}{2} - \frac{1}{7} - \frac{1}{8} = \frac{3}{2} - \frac{1}{7} - \frac{1}{8} = \frac{84}{56} - \frac{8}{56} - \frac{7}{56} = \boxed{\frac{69}{56}}$$

$$S_N = \frac{3}{2} - \frac{1}{N+1} - \frac{1}{N+2}$$

$$\lim_{n\to\infty} \frac{3}{2} - \frac{1}{N+1} - \frac{1}{N+2} = \lim_{x\to\infty} \frac{3}{2} - \frac{1}{x+1} - \frac{1}{x+2} = \frac{3}{2},$$

so $\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+2}\right)$ $\boxed{\text{converges to } \frac{3}{2}}$

Problem 4.

Prove the convergence or divergence of the following series using any rules or tests that work. You should verify the hypotheses of any test that you use carefully. In particular, inequalities and limit calculations should be justified.

4 (a) [4pts.] $\sum_{n=1}^{\infty} \frac{1}{n5^n}$

4 (b) [4pts.] $\sum_{n=1}^{\infty} \frac{4^n+n^2}{2^n-n}$

4 (c) [4pts.] $\sum_{n=1}^{\infty} \frac{n^3}{\sqrt{n^7+3n^2}}$.

1 (d) [4pts.] $\sum_{n=1}^{\infty} \frac{1}{100(\ln(n)+\sqrt{n})}$.

a. $\lim_{n\to\infty} \frac{1}{n5^n} = 0$

$0 \leq \frac{1}{n5^n} \leq \frac{1}{5^n}$ because $\frac{1}{n5^n}$ has a larger denominator than $\frac{1}{5^n}$.

$\sum_{n=1}^{\infty} \frac{1}{5^n} = \sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^n$ converges because $|r| = \frac{1}{5} < 1$ by the geometric series. ✓

Therefore, $\boxed{\sum_{n=1}^{\infty} \frac{1}{n5^n} \text{ converges by the direct comparison test}}$

b. $\lim_{n\to\infty} \frac{4^n+n^2}{2^n-n}$

$a_n = \frac{4^n+n^2}{2^n-n}$, $b_n = \frac{4^n}{2^n} = \left(\frac{4}{2}\right)^n = 2^n$

exponents of n grows quicker than 2

$\lim_{n\to\infty} \frac{4^n+n^2}{2^n(2^n-n)} = \frac{4^n+n^2}{2^{2n}-2^n n} = \frac{(2^2)^n+n^2}{(2^2)^n+2^n n} = \frac{1+\frac{n^2}{4^n}}{1+\frac{n}{2^n}} = \frac{1+0}{1+0} = 1$

$1 \neq 0, \infty$; $\sum_{n=1}^{\infty} 2^n$ diverges, so $\boxed{\sum_{n=1}^{\infty} \frac{4^n+n^2}{2^n-n} \text{ diverges by the limit comparison test}}$ ✓

c. $b_n = \frac{n^3}{n^{7/2}} = \frac{n^{6/2}}{n^{7/2}} = \frac{1}{n^{1/2}}$ ✓

$a_n = \frac{n^3}{\sqrt{n^7+3n^2}}$

$\lim_{n\to\infty} \frac{\sqrt{n}\, n^3}{\sqrt{n^7+3n^2}} = \frac{n^{7/2}}{\sqrt{n^7+3n^2}} = \frac{1}{1+\frac{\sqrt{3n^2}}{\sqrt{n^7}}} = \frac{1}{1+0} = 1 \neq 0, \infty$ ✓

$\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$, $p = \frac{1}{2} \leq 1$ $\therefore$ diverges by p-test.

so $\boxed{\sum_{n=1}^{\infty} \frac{n^3}{\sqrt{n^7+3n^2}} \text{ diverges by limit comparison test}}$ ✓

(d on last page)

d. $\lim_{n\to\infty} \frac{1}{100(\ln(n+\sqrt{n}))} = 0$

$a_n = $

$b_n = \frac{1}{100\sqrt{n}}$ $\}+1$

$\lim_{n\to\infty} \frac{100\sqrt{n}}{100(\ln(n)+\sqrt{n})} = \frac{\sqrt{n}}{\ln(n)+\sqrt{n}} = \frac{e^{\sqrt{n}}}{e^{\ln(n)+\sqrt{n}}} = \frac{e^{\sqrt{n}}}{1+e^{\sqrt{n}}}$

$= \frac{1}{1+\frac{1}{e^{\sqrt{n}}}} = 1 \neq 0, \infty$

?

~~$\sum_{n=1}^{\infty} \frac{1}{100\sqrt{n}}$ converges because $r = \frac{1}{100} < 1$~~

by geoseries

therefore $\sum_{n=1}^{\infty} \frac{1}{100(\ln(n)+\sqrt{n})}$ converges

by lim comp test

$a_n = \frac{1}{100(\ln(n)+\sqrt{n})}$ $\quad b_n = \frac{1}{100\sqrt{n}}$ $\quad b_n = \frac{1}{100\ln n}$

$\frac{100\sqrt{n}}{100(\ln(n)+\sqrt{n})} = \frac{\sqrt{n}}{\ln n+\sqrt{n}} = \frac{1}{\frac{\ln n}{\sqrt{n}}+1}$ $\quad \frac{e^{\sqrt{n}}}{e^{\ln n+\sqrt{n}}} = \frac{\cancel{e^{\sqrt{n}}}}{e^{\ln n}\cancel{e^{\sqrt{n}}}}$

$= \frac{1}{e^{\ln n}} = \frac{1}{n}$

$\frac{\ln n+\sqrt{n}}{\sqrt{n}} = \frac{\ln n}{\sqrt{n}}+1$

$= 0$

$\frac{\ln n}{\ln n+\sqrt{n}}$ $\quad \frac{\cancel{e^{(\ln n)}}^{\,n}}{e^{\sqrt{n}}} = \frac{1}{e^{\sqrt{n}}} = 0$