20W-MATH31B-1 Midterm 2

TOTAL POINTS

30 / 40

QUESTION 1

1 Problem 1(a) 1 / 3

- **- 0 pts** Correct
- **- 1.5 pts** Used formula for series starting at $$n=0$$
- ✓ **- 1 pts One error in $$r$$**
- **- 1.5 pts** Two errors in $$r$$
- ✓ **- 1 pts Error in $$c$$ or the first term**
- **- 0.5 pts** Computation mistake
- **- 3 pts** Incorrect or not attempted
- **- 2 pts** Correct solution, unclear procedure

QUESTION 2

2 Problem 1(b) 3 / 3

- ✓ **+ 1 pts $$|S-S_N| < b_{N+1}$$**
- ✓ **+ 0.75 pts Substitute correct $${N}$$**
- ✓ **+ 1.25 pts Substitute value of $$b_{N+1}$$**
- **- 0.5 pts** Confused $$S_N$$ with $$b_{N}$$
- **- 1 pts** Wrong index
- **- 1 pts** Use $$b_{N+1}$$ negative
- **+ 0 pts** Incorrect or not attempted
- **+ 2 pts** Correct solution not using alternating series bound

QUESTION 3

3 Problem 1(c) 3 / 3

- ✓ **- 0 pts Correct**
- **- 1 pts** Wrong ratio
- **- 1 pts** Does not mention $$r$$
- **- 2 pts** Serious flaws
- **- 3 pts** Incorrect or not attempted

QUESTION 4

4 Problem 2(a) 4 / 4

- ✓ **+ 4 pts Correct**
- **+ 1 pts** Converges
- **+ 1 pts** Root/ratio test attempt
- **+ 2 pts** Correctly did test
- **+ 0 pts** No credit

QUESTION 5

5 Problem 2(b) 0 / 4

- **+ 4 pts** Correct
- **+ 1 pts** Diverges
- **+ 2 pts** Divergence test
- **+ 1 pts** Limit of term in sum does not exist
- ✓ **+ 0 pts No credit**

QUESTION 6

6 Problem 2(c) 4 / 4

- ✓ **+ 4 pts Correct**
- **+ 1 pts** DCT/LCT attempt
- **+ 1 pts** Correct limit/inequality
- **+ 2 pts** Correctly used test to conclude divergence
- **+ 0 pts** No credit

QUESTION 7

7 Problem 3(a) 3 / 3

- **- 1 pts** Minor Mistake
- **- 2 pts** Only Show Efforts
- **- 3 pts** Blank or Incorrect
- ✓ **- 0 pts Correct**

QUESTION 8

8 Problem 3(b) 5 / 6

- ✓ **+ 1 pts Correctly substitute b=-0.1 and a=0 in the error bound inequality**
- ✓ **+ 1 pts Find a correct K with some but enough reasoning**
- **+ 2 pts** Correctly find a K with good reasoning
- ✓ **+ 1 pts Write inequality correctly to find N**
- ✓ **+ 2 pts Guess a correct N and justify it**

+ **1 pts** Simplify the error bound inequality by getting rid of (N+1)!

+ **1 pts** Use 10^(-10) to solve N

+ **1 pts** Correctly Solve for N

+ **0 pts** Blank or Complete Incorrect

QUESTION 9

9 Problem 4 **7 / 10**

- **0 pts** Correct with a good solution

- **1 pts** Minor errors like missing absolute value bars, sketchy justification

- **6 pts** Concluding the initial limit is 0 with consistent conclusion

✓ **- 3 pts Incorrect argument for upper endpoint**

- **0.5 pts** Not justifying lim (ln n)^(1/n)

- **0.5 pts** Missing Sigmas or incorrect use of Sigmas

- **8 pts** Mostly entirely incorrect but with a couple vaguely correct ideas

- **10 pts** Entirely incorrect

- **3 pts** Incorrect argument for lower endpoint

Math 31B
Integration and Infinite Series

Midterm 2

Instructions: You have 50 minutes to complete this exam. There are 4 questions, worth a total of 40 points. This test is closed book and closed notes. No calculator is allowed.

For full credit show all of your work legibly and justify your answers.

Please write your solutions in the space below the questions. If you run out of space, please *do not write on the back of the page*, but continue on the extra sheets attached at the end of the booklet. Please INDICATE if you continue on the extra sheets.

Do not forget to write your name, section and UID legibly in the space below.

Name: ______________________________

Student ID number: ______________________________

Section (circle one):

	Tuesday	Thursday
Weiyi Liu	1A	1B
Timothy Smits	1C	1D
Joseph Breen	1E	1F

Question	Points	Score
1	9	
2	12	
3	9	
4	10	
Total:	40	

FORMULAE

The N-th Taylor polynomial of $f(x)$ at a is given by

$$\sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Error bound theorem.
Let $T_N(x)$ be the N^{th} Taylor polynomial centered at $a \in \mathbb{R}$ associated to $f(x)$. Suppose that $\left|f^{N+1}(u)\right| \leq K$ for all u between a and b. Then

$$|f(b) - T_N(b)| \leq K \, \frac{|b-a|^{N+1}}{(N+1)!}$$

Trigonometric identities

$$\cos^2(x) + \sin^2(x) = 1$$

$$\begin{aligned}\sin(2x) &= 2\sin(x)\cos(x)\\ \cos(2x) &= \cos^2(x) - \sin^2(x)\end{aligned}$$

$$\begin{aligned}\sin^2(x) &= \frac{1}{2}(1 - \cos(2x))\\ \cos^2(x) &= \frac{1}{2}(1 + \cos(2x))\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}\sin(x) &= \cos(x)\\ \frac{d}{dx}\cos(x) &= -\sin(x)\end{aligned}$$

Problem 1.

(a) **[3pts.]** Find the *exact value* S to which the series $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot 10}{8^n}$ converges.

[Note that the series starts at $n = 1$ and not at $n = 0$.]

Geometric Series:

if $|r| < 1$, converges to $\frac{cr^k}{1-r}$

Prove absolute convergence.

$$\sum_{n=1}^{\infty} \frac{(-1)^n \cdot 10}{8^n}$$

$$a_n = \frac{(-1)^n \cdot 10}{8^n}$$

Absolute convergence!

$$a_n: \left| \frac{\cancel{(-1)^n} \cdot 10}{8^n} \right|$$

$$a_n = 10 \cdot \left(\frac{1}{8}\right)^n$$

$r = \frac{1}{8} < 1$, so converges.

Since it is a geometric series, it converges to:

$$\boxed{\frac{\frac{10}{8}}{1 - \frac{1}{8}}}$$

(b) **[3pts.]** Using your knowledge of alternating series, estimate the error $|S - S_8|$, where S_8 denotes the 8-th partial sum of the series in part (a), i.e. $S_8 = \sum_{n=1}^{8} \frac{(-1)^n \cdot 10}{8^n}$.

Alt. series test

$b_n = \frac{10}{8^n}$

- b_n is decreasing
- $\lim_{n \to \infty} b_n = 0$

Passes AST

Alt. ser. ET

$|S - S_N| \leq b_{N+1}$

$$\boxed{|S - S_8| \leq \frac{10}{8^9}}$$

$$\frac{10}{8^9}$$

(c) **[3pts.]** Does the series in part (a) converge absolutely or conditionally? Remember to justify your answer.

The series converges absolutely.

Series: $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot 10}{8^n}$, abs. conv. can be found w/ absolute value.

$\sum_{n=1}^{\infty} \left| \frac{10}{8^n} \right|$ this is a geometric series with $r = \frac{1}{8}$.

$\frac{1}{8} < 1$, so by the rules of geometric series, it converges;

so it converges absolutely.

Problem 2.

For each of the following series, state if it converges or if it diverges. Justify your answers carefully.

(a) [4pts.] $\sum_{n=1}^{\infty} \frac{4^n \cdot n^4}{(\sqrt[4]{n})^n}$

Root test:

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{4^n \cdot n^4}{(n^{1/4})^n}\right|}$$

$$\lim_{n\to\infty} \frac{4 \cdot \sqrt[n]{n}^4}{n^{1/4}} = \lim_{n\to\infty} \frac{4 \cdot 1^4}{n^{1/4}} = \lim_{n\to\infty} \frac{4}{n^{1/4}} = 0$$

$0<1$, so by the root test, $\sum_{n=1}^{\infty} \frac{4^n \cdot n^4}{(\sqrt[4]{n})^n}$ converges.

(b) [4pts.] $\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{1}{n^2}\right)$

Prove absolute convergence:

If $\sum_{n=1}^{\infty} \left|\cos\left(\frac{1}{n^2}\right)\right|$ converges, $\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{1}{n^2}\right)$ converges absolutely.

Direct comparison test:

$b_n = \frac{1}{n^2}$ $\left|\cos\left(\frac{1}{n^2}\right)\right| \leq \frac{1}{n^2}$

$0 \leq \left|\cos\left(\frac{1}{n^2}\right)\right| \leq \frac{1}{n^2}$

$\sum \frac{1}{n^2}$ converges by p-test, so by direct comparison test, $\sum \left|\cos\left(\frac{1}{n^2}\right)\right|$ converges.

Therefore, $\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{1}{n^2}\right)$ converges.

(c) [4pts.] $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n} \cdot \ln n}$

Limit comparison test:

$b_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}\ln n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{\sqrt{n}\ln n} = \lim_{n\to\infty} \frac{n^{1/2}}{\ln n}$$

L'Hopitals

$$\lim_{n\to\infty} \frac{\frac{1}{2}n^{-1/2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{2\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{2} = \infty$$

By LCT, a_n is a larger series, b_n is smaller.

$\sum \frac{1}{n}$ diverges by the p-test $(1 \leq 1)$, so by the LCT,

$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n} \cdot \ln n}$ diverges as well.

Problem 3.

Let $f(x) = e^x$.

(a) [**3pts.**] Write the 4th Maclaurin polynomial $T_4(x)$.
[You do not need to justify your answer if you remember it from lecture.]

$$\sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n \qquad \sum_{n=0}^{4} \frac{f^{(n)}(0)}{n!}(x)^n$$

$f^{0}(x) = e^x$

$f'(x) = e^x$

$f''(x) = e^x$

$f'''(x) = e^x$

$f^{4}(x) = e^x$

$$T_4(x) = \frac{e^0}{0!} + \frac{e^0 x}{1!} + \frac{e^0 x^2}{2!} + \frac{e^0 x^3}{3!} + \frac{e^0 x^4}{4!}$$

$$\boxed{T_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}}$$

(b) [6pts.] Find a number N such that $T_N(-0.1)$, the N-th Maclaurin polynomial computed at $b = -0.1$, approximates $e^{-0.1}$ with an error less than 10^{-10}.

$$|f(b) - T_N(b)| \le K \frac{|b-a|^{N+1}}{(N+1)!}$$

$$|f^{N+1}(u)| \le K \qquad u = [a,b]$$

$0, -0.1$

0 is greater

$$|e^{-0.1} - T_N(-0.1)| \le K \frac{|-0.1|^{N+1}}{(N+1)!}$$

Always will be:

e^x

$e^0 = 1 \qquad K = 1$

$$|e^{-0.1} - T_N(-0.1)| \le \frac{0.1^{N+1}}{(N+1)!}$$

$$\frac{0.1^{N+1}}{(N+1)!} \le \frac{1}{10^{10}}$$

$$\boxed{N = 10^{10}}$$

$$\frac{\overbrace{0.1^{10^{10}+1}}^{<1}}{\underbrace{(10^{10}+1)!}_{>10^{10}}} \le \frac{1}{10^{10}}$$

Problem 4. *10pts.*

Find the interval of convergence of the series $\sum_{n=2}^{\infty} \frac{(x-1)^n}{2^n \cdot \ln n}$.

Ratio test

$$\lim_{n\to\infty} \left| \frac{(x-1)^{n+1}}{2^{n+1}\cdot\ln(n+1)} \cdot \frac{2^n\cdot\ln(n)}{(x-1)^n} \right|$$

$$\lim_{n\to\infty} \left| \frac{(x-1)\cancel{(x-1)^n}}{2\cdot\cancel{2^n}\cdot\ln(n+1)} \cdot \frac{\cancel{2^n}\cdot\ln(n)}{\cancel{(x-1)^n}} \right|$$

$$\lim_{n\to\infty} \frac{|(x-1)|}{2} \cdot \frac{\ln(n)}{\ln(n+1)}$$

$$\frac{|x-1|}{2} \cdot \lim_{n\to\infty} \frac{\ln(n)}{\ln(n+1)}$$

L'Hopitals

$$\frac{|x-1|}{2} \cdot \lim_{n\to\infty} \frac{n+1}{n} = \frac{|x-1|}{2}$$

goes to 1

Converges if:

$$\frac{|x-1|}{2} < 1$$

$$|x-1| < 2$$

$$-2 < x-1 < 2$$

$$-1 < x < 3$$

Check endpoints:

$-1 < x < 3$

$x = -1$

$$\sum_{n=2}^{\infty} \frac{(-2)^n}{2^n \cdot \ln n}$$

$$\sum_{n=2}^{\infty} \frac{(-1)^n \cancel{(2)^n}}{\cancel{2^n}\cdot\ln n}$$

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$$

By AST:

$b_n = \frac{1}{\ln n}$

- b_n is decreasing
- $\lim_{n\to\infty} b_n = 0$

So, $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ converges,

Therefore, $\sum_{n=2}^{\infty} \frac{(x-1)^n}{2^n \ln n}$ converges at $x=-1$.

$x = 3$

$$\sum_{n=2}^{\infty} \frac{\cancel{(2)^n}}{\cancel{2^n}\ln n}$$

$$\sum_{n=2}^{\infty} \frac{1}{\ln n}$$

LCT: $b_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{\frac{1}{\ln n}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{\ln n}$$

L'Hopitals

$$\lim_{n\to\infty} \frac{2n}{\frac{1}{n}} = \lim_{n\to\infty} 2n^2$$

$$= \infty$$

By LCT, b_n is a greater series

$\sum \frac{1}{n^2}$ converges by P-test $(2 > 1)$, so by LCT, $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ also converges.

Therefore, $\sum_{n=2}^{\infty} \frac{(x-1)^n}{2^n \ln n}$ converges at $x = 3$.

Therefore, the interval of convergence is:

$$[-1, 3]$$

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