20W-MATH31B-1 Final

SANCHIT AGARWAL

TOTAL POINTS

99 / 100

QUESTION 1

1 Problem 1.(a) 5 / 5

✓ **- 0 pts** Correct

- 1 pts One of them is incorrect.

- 2 pts Two of them are incorrect.

- 3 pts Three of them are incorrect.

- 4 pts Four of them are incorrect.

- 5 pts Five of them are incorrect.

QUESTION 2

2 Problem 1.(b) 5 / 5

✓ **- 0 pts** Correct

- 1 pts One of them is incorrect.

- 2 pts Two of them are incorrect.

- 3 pts Three of them are incorrect.

- 4 pts Four of them are incorrect.

- 5 pts Five of them are incorrect.

QUESTION 3

3 Problem 2 10 / 10

✓ **- 0 pts** Correct

- 0 pts Significant mishandling of the 1-cos(t^2) / t^4 step

- 0 pts Mishandling of the 1-cos(t^2) / t^4 step

- 0 pts Incorrect Sigma algebraic manipulation

- 0 pts Not canceling out n=0 term correctly

- 0 pts Incorrect pattern recognition for final Sigma expression

- 0 pts Incorrect / misunderstanding the question

- 0 pts No sigma notation in final answer

- 0 pts 10/10 by default

Some of the notation is a little imprecise but okay

QUESTION 4

4 Problem 3 5 / 5

✓ **- 0 pts** Correct

- 4 pts Considered $$\sum \left|\frac{(-1)^n}{n^4+\sqrt n}\right|$$ to show absolute convergence, but reasoning for $$\sum \frac1{n^4 +\sqrt n}$$ is completely wrong or missing.

- 3 pts Showed convergence of $$\sum \frac{(-1)^n}{n^4+\sqrt n}$$ by AST, but reasoning for the series of absolute values (if any) is completely incorrect.

- 2 pts Structure is correct but application of LCT/DCT is incorrect.

- 1 pts Algebra error

- 0.5 pts Minor mistake (e.g., improper use of $$|\cdot|$$ or $$\sum$$)

- 5 pts Completely incorrect

QUESTION 5

5 Problem 4.(a) 5 / 5

+ 2 pts Correct derivative

+ 2 pts Show that f'(x) > 0 on (-infinity,0) and (0,infinity)

+ 1 pts Deduce that f is strictly increasing, so invertible

✓ **+ 5 pts All correct**

+ 4 pts Forgot that f'(0) = 0.

+ 0 pts No credit

QUESTION 6

6 Problem 4.(b) 5 / 5

+ 2 pts Computed g(pi/4).

+ 2 pts Used formula to compute g'(pi/4).

+ 1 pts Correct answer

✓ **+ 5 pts All correct**

+ 0 pts No credit

QUESTION 7

7 Problem 4.(c) 4 / 5

✓ + **2 pts** **Correct Maclaurin series**

✓ + **2 pts** **Compared coefficients with general Taylor series**

+ **1 pts** Correct answers

+ **5 pts** Correct

+ **0 pts** No credit

f^{(11)}(0) = -11!/7, this is a number, it can't equal some polynomial.

QUESTION 8

8 Problem 5.(a) 5 / 5

✓ - **0 pts** **Correct**

- **1 pts** Computation mistakes (e.g. wrong value for the derivatives) or typo in the final solution
- **2 pts** More computation mistakes
- **2 pts** The polynomial is not centered at pi/2 in the final answer.
- **2.5 pts** The answer is not a polynomial (e.g. you do not substitute the value of the derivatives at the point).
- **2.5 pts** The polynomial has the wrong degree or misses the some terms.
- **4 pts** Lack of justification.
- **5 pts** Wrong

QUESTION 9

9 Problem 5.(b) 5 / 5

✓ - **0 pts** **Correct**

- **1 pts** Claim |-cos u| = cos u without saying which interval u is in.
- **1 pts** Correct K but missing or incomplete justification (e.g. did not check monotonicity of |-cos(x)| or did not claim |cos(x)| is bounded by 1). Claiming |Sin(x)| is bounded by 1 is not enough.
- **2 pts** Wrong K or Ignored K
- **2 pts** Wrong error bound formula
- **2 pts** Wrong |b-a|^3 (e.g. use |pi/2-0.1|^3 rather than 0.1^3).
- **5 pts** Wrong or blank.

QUESTION 10

10 Problem 6 10 / 10

✓ - **0 pts** **Correct**

- **0.5 pts** Minor mistakes (e.g., absolute value bars)
- **1 pts** Incorrect use of $$\Sigma$$
- **0.5 pts** Not properly justifying $$\lim_{n \to +\infty} \frac{\sqrt n}{\sqrt{n+1}} =1$$
- **2.5 pts** Wrong reasoning for $$x=-5$$
- **2.5 pts** Wrong reasoning for $$x=1$$
- **1 pts** Not checking conditions of AST
- **1 pts** Mistake in computing limit in ratio/root test
- **2.5 pts** Ratio/root test cannot work again at endpoints
- **1 pts** Not justifying $$\lim_{n\to+\infty} \sqrt[n]{\sqrt n} = 1$$

QUESTION 11

11 Problem 7 10 / 10

✓ - **0 pts** **Correct**

- **0 pts** Click here to replace this description.
- **10 pts** Incorrect

QUESTION 12

12 Problem 8.(a) 8 / 8

✓ - **0 pts** **Correct**

- **2 pts** Algebraic simplification error after the trig sub
- **0.5 pts** Minor error (e.g. forgetting a constant or something, incorrect reduction formula)
- **1.5 pts** Lack of explanation for integrating sin^2(theta)
- **3 pts** Significantly incorrect integration
- **2.5 pts** Leaving final answer in terms of theta
- **3 pts** Trig sub implemented incorrectly (no differential computed, etc.)

QUESTION 13

13 Problem 8.(b) 8 / 8

✓ - **0 pts** **Correct**

- **1 pts** Forget writing the absolute sign in ln|x| or ln|x+1|
- **1 pts** Forget adding constant C to the indefinite

integral

- **1 pts** Computation mistakes.
- **2 pts** More Computation Mistakes
- **3 pts** Wrong Factorization at the first place
- **4 pts** Omit some important steps in factorization or finding A, B, C.
- **6 pts** Find A, B, and C (partial fraction parts) incorrectly or without justification.
- **8 pts** Completely Wrong
- **4 pts** Incomplete or did not find the indefinite integral to the end.

QUESTION 14

14 Problem 9.(a) 7 / 7

✓ **- 0 pts Correct**

- **1 pts** Missing C
- **1 pts** Missing $$| \cdot |$$
- **1 pts** Algebra error
- **2 pts** Multiple algebra errors
- **4 pts** Conceptual problem(s) with substitution or integration by parts
- **7 pts** Totally incorrect or not attempted

QUESTION 15

15 Problem 9.(b) 7 / 7

✓ **- 0 pts Correct**

- **5 pts** The solution does not show the computation of $$\lim_{R\to0^+} (-2\sin R \cdot \ln(\sin R) +2\sin R)$$
(or analogous formula with $$\ln(\sin^2 R)$$)
- **4 pts** Did not realise/write that $$\lim_{R\to0^+} \sin R \cdot \ln(\sin R) = 0 \cdot \infty$$
(or analogously with $$\ln(\sin^2 R)$$)
- **3 pts** Realised that $$\lim_{R\to0^+} \sin R \cdot \ln(\sin R) = 0 \cdot \infty$$ but did not write that it is an indeterminate form
- **1 pts** Forgot $$\cos x$$ in the derivative of $$\frac1{\sin x}$$
- **3 pts** Error in part (a) makes problem solvable without L'Hopital's rule
- **1 pts** Algebra error
- **7 pts** Blank or completely incorrect (e.g., limit at wrong endpoint, does not use the definition of improper integral)

Problem 1.

For the next two questions, mark all correct answers (they can be any number between 0 and 5). No justification is required.

(a) [5pts.] The series $\sum_{n=0}^{\infty}(-1)^n \cdot \frac{1}{3^n}$ converges (*mark all correct answers, no justification required*)

- ◯ by the root test
- ◯ by the ratio test
- ◯ by limit comparison test with $\sum \frac{1}{n}$
- ◯ by the alternating series test
- ◯ because it is a p-series with $p = -\frac{1}{3}$

1 Problem 1.(a) **5 / 5**

- ✓ **- 0 pts** **Correct**
- **- 1 pts** One of them is incorrect.
- **- 2 pts** Two of them are incorrect.
- **- 3 pts** Three of them are incorrect.
- **- 4 pts** Four of them are incorrect.
- **- 5 pts** Five of them are incorrect.

(b) [5pts.] The series $\sum_{n=1}^{\infty} (-1)^n \cdot \left(1 + \frac{1}{n}\right)^{-n}$ diverges (*mark all correct answers, no justification required*)

- ○ by the alternating series test
- ○ because it is a geometric series with $|r| \geq 1$
- ○ by absolute convergence
- ✓ by the divergence test
- ○ by the root test

2 Problem 1.(b) **5 / 5**

- ✓ **- 0 pts** Correct
- **- 1 pts** One of them is incorrect.
- **- 2 pts** Two of them are incorrect.
- **- 3 pts** Three of them are incorrect.
- **- 4 pts** Four of them are incorrect.
- **- 5 pts** Five of them are incorrect.

Problem 2. *10pts.*

Find the Maclaurin series of $f(x) = \int_0^x \frac{1-\cos(t^2)}{t^4}\mathrm{d}t$.

For full credit, please write it using the compact series notation, i.e., $\sum_{n=\cdots}^{\cdots} \cdots$.

$$f(x) = \int_0^x \frac{1-\cos(t^2)}{t^4} dt$$

$$\text{as } \cos(t^2) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n!}$$

[KNOWN MACLAURIN SERIES]

$$= \int_0^x \frac{1 - \left[1 - \frac{t^4}{2!} + \frac{t^8}{4!} - \frac{t^{12}}{6!} \cdots\right]}{t^4}$$

$$= \int_0^x \frac{\frac{t^4}{2!} - \frac{t^8}{4!} + \frac{t^{12}}{6!} \cdots}{t^4}$$

$$= \int_0^x \frac{t^0}{2!} - \frac{t^4}{4!} + \frac{t^8}{6!} - \frac{t^{12}}{8!} \cdots$$

$$= \int_0^x \sum_{n=1}^{\infty} \frac{(-1)^{n-1} t^{4n-4}}{2n!}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{4n-3}}{(4n-3)\, 2n!}$$

3 Problem 2 **10 / 10**

- ✓ **- 0 pts** Correct
- **- 0 pts** Significant mishandling of the 1-cos(t^2) / t^4 step
- **- 0 pts** Mishandling of the 1-cos(t^2) / t^4 step
- **- 0 pts** Incorrect Sigma algebraic manipulation
- **- 0 pts** Not canceling out n=0 term correctly
- **- 0 pts** Incorrect pattern recognition for final Sigma expression
- **- 0 pts** Incorrect / misunderstanding the question
- **- 0 pts** No sigma notation in final answer
- **- 0 pts** 10/10 by default

Some of the notation is a little imprecise but okay

Problem 3. *5pts.*

Does the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4+\sqrt{n}}$ converge absolutely, conditionally, or not at all?

$$\Sigma a_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4+\sqrt{n}}$$

let's first evaluate $\Sigma |a_n|$

$$\Sigma |a_n| = \Sigma a_n' = \sum_{n=1}^{\infty} \frac{1}{n^4+\sqrt{n}}$$

Now applying limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^2} = \Sigma b_n$

$$\therefore L = \lim_{n\to\infty} \frac{b_n}{a_n'}$$

$$\Rightarrow L = \lim_{n\to\infty} \frac{n^4+\sqrt{n}}{n^2}$$

$$\Rightarrow L = \lim_{n\to\infty} (n^2 + n^{-3/2})$$

$$\Rightarrow L = \infty$$

NOTE: I AM USING SELF TAKEN NOTATION BUT MEANING IS SAME

$\therefore$ By limit comparision test, when $L=\infty$ & Σb_n converges

$\Rightarrow \Sigma a_n'$ converges

Now by Absolute convergence as $\Sigma a_n' = \Sigma |a_n|$ converges

$\Rightarrow \Sigma a_n$ converges Absolutely

4 Problem 3 **5 / 5**

✓ **- 0 pts** **Correct**

- 4 pts Considered $$\sum \left|\frac{(-1)^n}{n^4+\sqrt n}\right|$$ to show absolute convergence, but reasoning for $$\sum \frac1{n^4 +\sqrt n}$$ is completely wrong or missing.

- 3 pts Showed convergence of $$\sum \frac{(-1)^n}{n^4+\sqrt n}$$ by AST, but reasoning for the series of absolute values (if any) is completely incorrect.

- 2 pts Structure is correct but application of LCT/DCT is incorrect.

- 1 pts Algebra error

- 0.5 pts Minor mistake (e.g., improper use of $$|\cdot|$$ or $$\sum$$)

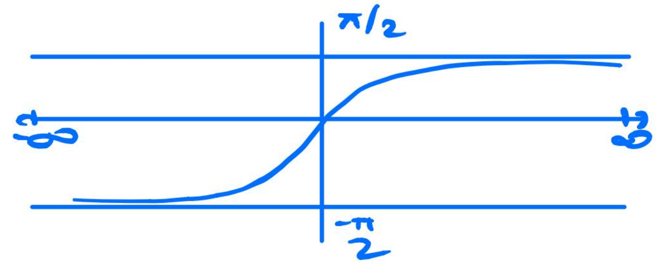
- 5 pts Completely incorrect

Problem 4.

Let $f(x) = x^4 \cdot \tan^{-1} x$, defined on $(-\infty, +\infty)$.

(a) [5pts.] Show that $f(x)$ is invertible.

tan⁻¹x



$f(x) = x^4 \cdot \tan^{-1} x$

Now

$f'(x) = 4x^3 \tan^{-1} x + \dfrac{x^4}{1+x^2}$

→ For all $x \in (-\infty, \infty)$

$x^4 > 0$

$x^2 > 0$

$\Rightarrow \dfrac{x^4}{1+x^2} > 0$ — (1)

→ Now for all $x \in (0, +\infty)$

$\tan^{-1} x > 0$ [see graph above]

$x^3 > 0$

$\Rightarrow 4x^3 \tan^{-1} x > 0$ — (2)

By (1) & (2)

$\Rightarrow f'(x) > 0$

→ Now for all $x \in (-\infty, 0)$

$\tan^{-1} x < 0$ [see graph]

$x^3 < 0$

$\Rightarrow 4x^3 \tan^{-1} x > 0$ — (3)

By (1) & (3)

$\Rightarrow f'(x) > 0$

→ Now at $x = 0$

$f'(x) = 0 + 0 = 0$

$\therefore$ for all $x \in (-\infty, \infty)$

$f'(x) \geq 0$

$\Rightarrow f(x)$ is increasing

$\Rightarrow f(x)$ is one-one in its domain

$\Rightarrow$ **$f(x)$ is invertible**

5 Problem 4.(a) **5 / 5**

- **+ 2 pts** Correct derivative
- **+ 2 pts** Show that f'(x) > 0 on (-infinity,0) and (0,infinity)
- **+ 1 pts** Deduce that f is strictly increasing, so invertible
- ✓ **+ 5 pts** **All correct**
- **+ 4 pts** Forgot that f'(0) = 0.
- **+ 0 pts** No credit

(b) [5pts.] If $g(x)$ denotes the inverse to $f(x)$, find $g(\pi/4)$ and $g'(\pi/4)$.

Now,

$g(x)$ is the inverse of $f(x)$

as we know by property of inverse

$f(g(x)) = x$

differentiating both sides w.r.t x

$f'(g(x)) \times g'(x) = 1$

$\Rightarrow g'(x) = \frac{1}{f'(g(x))}$ — ①

Now by observation

$f(1) = 1 \times \tan^{-1}(1) = \frac{\pi}{4}$

$\Rightarrow g(\frac{\pi}{4}) = 1$

By ①

$g'(\frac{\pi}{4}) = \frac{1}{f'(g(\frac{\pi}{4}))} = \frac{1}{f'(1)}$

$g'(\frac{\pi}{4}) = \frac{1}{4(1)^3 \tan^{-1} 1 + \left(\frac{1^4}{1+1^2}\right)} = \frac{1}{4 \times \frac{\pi}{4} + \frac{1}{2}}$

$= \boxed{\frac{2}{2\pi + 1}}$

6 Problem 4.(b) **5 / 5**

- \+ **2 pts** Computed g(pi/4).
- \+ **2 pts** Used formula to compute g'(pi/4).
- \+ **1 pts** Correct answer
- ✓ + **5 pts** **All correct**
- \+ **0 pts** No credit

(c) [5pts.] Find $f^{(11)}(0)$ and $f^{(12)}(0)$.

$$f(x) = x^4 \tan^{-1} x$$

$$f(x) = x^4 \cdot \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+5}}{2n+1}$$

Now as we know as a matter of fact that any taylor expansion is of the form

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a) x^n}{n!} = f(a) + \frac{f'(a)x}{1!} + \frac{f^{(2)}(a)x^2}{2!} \dots \dots$$

$\therefore$ as we notice that the coefficient of any x^p is $\frac{f^p(a)}{p!}$ — ①

Now to find $f^{11}(0)$

lets evaluate for $p = 11$

$\Rightarrow 2n+5 = 11$

$\Rightarrow n = 3$

$\therefore$ coefficient of x^{11} is $\frac{(-1)^3}{2(3)+1}$

$= \frac{-1}{7}$ —— (2)

By (1) & (2)

$$\frac{f^{(11)}(0)}{11!} = \frac{-1}{7}$$

$$f^{(11)}(0) = \frac{-11!}{7} = \left(\frac{-1}{7}\right) \times 11!$$

Now for $p = 12$

$2n + 5 = 12$

$\therefore$ there exist no such integral n for $p = 12$

$\Rightarrow$ coefficient of x^{12} must be zero

Ans $\rightarrow$ $f^{(11)}(0) = \left(\frac{-1}{7}\right) \times 11!$

$f^{(12)}(0) = 0$

7 Problem 4.(c) **4 / 5**

- ✓ **+ 2 pts** **Correct Maclaurin series**
- ✓ **+ 2 pts** **Compared coefficients with general Taylor series**
- **+ 1 pts** Correct answers
- **+ 5 pts** Correct
- **+ 0 pts** No credit

f^{(11)}(0) = -11!/7, this is a number, it can't equal some polynomial.

Problem 5.

Let $f(x) = \sin x$, and $a = \frac{\pi}{2}$.

(a) [5pts.] Find $T_2(x)$, the 2-nd Taylor polynomial of $f(x)$ at $a = \frac{\pi}{2}$. Show your work.

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

$$T_2(x) = \frac{f(c)}{0!} + \frac{f^{(1)}(c)(x-c)}{1!} + \frac{f^{(2)}(c)(x-c)^2}{2!}$$

Now here

$f(x) = \sin x$

$a = \frac{\pi}{2}$

$$T_2(x) = 1 + \frac{\cancel{\cos\left(\frac{\pi}{2}\right)}\left(x-\frac{\pi}{2}\right)}{1!} \to 0 - \frac{\cancel{\sin\left(\frac{\pi}{2}\right)}\left(x-\frac{\pi}{2}\right)^2}{2!}$$

$$T_2(x) = 1 - \frac{\left(x-\frac{\pi}{2}\right)^2}{2!}$$

$$T_2(x) = 1 - \frac{1}{2}\left(x^2 + \frac{\pi^2}{4} - \pi x\right)$$

$$T_2(x) = 1 - \frac{x^2}{2} - \frac{\pi^2}{8} + \frac{\pi x}{2}$$

8 Problem 5.(a) **5 / 5**

- ✓ **- 0 pts** **Correct**
- **- 1 pts** Computation mistakes (e.g. wrong value for the derivatives) or typo in the final solution
- **- 2 pts** More computation mistakes
- **- 2 pts** The polynomial is not centered at pi/2 in the final answer.
- **- 2.5 pts** The answer is not a polynomial (e.g. you do not substitute the value of the derivatives at the point).
- **- 2.5 pts** The polynomial has the wrong degree or misses the some terms.
- **- 4 pts** Lack of justification.
- **- 5 pts** Wrong

(b) [5pts.] Use the error bound theorem to estimate the error $\left|\sin\left(\frac{\pi}{2}-0.1\right)-T_2\left(\frac{\pi}{2}-0.1\right)\right|$. You do not need to simplify your answer as much as possible. However, your final answer must be a number, and should not contain letters.

By ERROR BOUND THEOREM

$$\left|\sin\left(\frac{\pi}{2}-0.1\right)-T_2\left(\frac{\pi}{2}-0.1\right)\right| < \frac{K}{(n+1)!}\left|\left(\frac{\pi}{2}+0.1\right)-\frac{\pi}{2}\right|^{n+1}$$

Now,

As we are dealing with T_2

$\Rightarrow n=2$

$$\Rightarrow \left|\sin\left(\frac{\pi}{2}-0.1\right)-T_2\left(\frac{\pi}{2}-0.1\right)\right| < \frac{K}{3!}\left|0.1\right|^3$$

$f^3(t) = -\cos t$

$\Rightarrow |f^3(t)| = \cos t$

Now as t approaches $\frac{\pi}{2}$, $\cos t$ decreases

$\Rightarrow |f^3(t)|$ in the range $t \in \left[\frac{\pi}{2}-0.1, \frac{\pi}{2}\right]$ is max

at $t = \frac{\pi}{2}-0.1$

Now as $K \geq |f^3(t)|$

$$K \geq \cos\left(\frac{\pi}{2} - 0.1\right)$$

$$K \geq 0.0998$$

$$\Rightarrow K \geq 0.1$$

$$\Rightarrow \left|\sin\left(\frac{\pi}{2} - 0.1\right) - T_2\left(\frac{\pi}{2} - 0.1\right)\right| = 0.1 \times \frac{1}{6 \times 1000}$$

$$\Rightarrow \text{ERROR} = 0.166 \times 10^{-4}$$

$$\Rightarrow \text{ERROR} = 1.667 \times 10^{-5}$$

9 Problem 5.(b) **5 / 5**

✓ **- 0 pts** **Correct**

- 1 pts Claim |-cos u| = cos u without saying which interval u is in.

- 1 pts Correct K but missing or incomplete justification (e.g. did not check monotonicity of |-cos(x)| or did not claim |cos(x)| is bounded by 1). Claiming |Sin(x)| is bounded by 1 is not enough.

- 2 pts Wrong K or Ignored K

- 2 pts Wrong error bound formula

- 2 pts Wrong |b-a|^3 (e.g. use |pi/2-0.1|^3 rather than 0.1^3).

- 5 pts Wrong or blank.

Problem 6. *10pts.*

Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(x+2)^n}{\sqrt{n} \cdot 3^n}$.

$$\sum_{n=1}^{\infty} \frac{(x+2)^n}{\sqrt{n} \cdot 3^n}$$

Now using ratio test

$$\Rightarrow \rho = \lim_{n \to \infty} \left| \frac{(x+2)^{n+1} \times \sqrt{n} \times 3^n}{\sqrt{n+1} \times 3^{n+1} \times (x+2)^n} \right|$$

$$\Rightarrow \rho = \lim_{n \to \infty} \left| \frac{(x+2)}{3} \sqrt{\frac{1}{1+1/n}} \right|$$

$$\Rightarrow \rho = \left| \frac{x+2}{3} \right|$$

Now for series to converge

$\rho < 1$

$$\Rightarrow \left| \frac{x+2}{3} \right| < 1$$

$$\Rightarrow |x+2| < 3$$

$$\Rightarrow -5 < x < 1$$

Now checking at endpoints

At $x=-5$

Series $\rightarrow \sum a_n = \sum_{n=1}^{\infty} \frac{(-3)^n}{\sqrt{n}\cdot 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

Now considering the sequence $\{a_n\}$

By alternating series test,

as a_n is decreasing

& $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, $N \geq 1$

$\Rightarrow$ the series converges

$\Rightarrow$ Power series converges at $x=-5$

At $x=1$

series $\rightarrow \sum b_n = \sum_{n=1}^{\infty} \frac{3^n}{\sqrt{n}\times 3^n} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

By p-series test

here $p = \frac{1}{2}$

$\Rightarrow p < 1$

$\Rightarrow$ series diverges

$\Rightarrow$ Power series diverges at $x=1$

[In series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ if $p<1 \Rightarrow$ series diverges]

$\therefore$ INTERVAL OF CONVERGENCE

$= [-5, 1)$

10 Problem 6 **10 / 10**

- ✓ **- 0 pts** Correct
- **- 0.5 pts** Minor mistakes (e.g., absolute value bars)
- **- 1 pts** Incorrect use of $$\Sigma$$
- **- 0.5 pts** Not properly justifying $$\lim_{n \to +\infty} \frac{\sqrt n}{\sqrt{n+1}} =1$$
- **- 2.5 pts** Wrong reasoning for $$x=-5$$
- **- 2.5 pts** Wrong reasoning for $$x=1$$
- **- 1 pts** Not checking conditions of AST
- **- 1 pts** Mistake in computing limit in ratio/root test
- **- 2.5 pts** Ratio/root test cannot work again at endpoints
- **- 1 pts** Not justifying $$\lim_{n\to+\infty} \sqrt[n]{\sqrt n} = 1$$

Problem 7. *10pts.*

Using integration by parts, derive the following reduction formula:

$$\int \cos^n x \, \mathrm{d}x = \frac{1}{n} \cos^{n-1} x \cdot \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, \mathrm{d}x$$

According to Integration by parts

$$\int f(x)g(x)\,dx = f(x)\int g(x)\,dx - \int\left(f'(x)\int g(x)\,dx\right)dx$$

Now

let $f(x) = \cos^{n-1}x$

$g(x) = \cos x$

$$\Rightarrow \int \cos^{n-1}x \cdot \cos x\,dx = \cos^{n-1}x\sin x + \int (n-1)\cos^{n-2}x \times \sin^2 x\,dx$$

$$\Rightarrow \int \cos^{n-1}x \cdot \cos x\,dx = \cos^{n-1}x\sin x + \int (n-1)\cos^{n-2}x\,(1-\cos^2 x)\,dx$$

$$\Rightarrow \int \cos^{n-1}x\cos x\,dx = \cos^{n-1}x\sin x + \int (n-1)\cos^{n-2}x\,dx - \int (n-1)\cos^n x\,dx$$

$$\Rightarrow \int \cos^n x\,dx + \int (n-1)\cos^n x\,dx = \cos^{n-1}x\sin x + \int (n-1)\cos^{n-2}x\,dx$$

$$\Rightarrow n\int \cos^n x\,dx = \cos^{n-1}x\sin x + \int (n-1)\cos^{n-2}x\,dx$$

$$\Rightarrow \int \cos^n x\,dx = \frac{\cos^{n-1}x}{n}\sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx$$

Hence Proved.

11 Problem 7 **10 / 10**

✓ **- 0 pts** **Correct**

- 0 pts Click here to replace this description.

- 10 pts Incorrect

Problem 8.

Evaluate the following integrals. Show your work.

(a) [8pts.] $\int \frac{x^2}{\sqrt{2-x^2}}\,\mathrm{d}x$.

Here,

Let $x = \sqrt{2}\sin\theta$

$dx = \sqrt{2}\cos\theta\, d\theta$

$$\Rightarrow \int \frac{2\sqrt{2}\sin^2\theta\cos\theta\, d\theta}{\sqrt{2-2\sin^2\theta}}$$

$$\Rightarrow \int \frac{2\sqrt{2}\sin^2\theta\cos\theta\, d\theta}{\sqrt{2}\sqrt{1-\sin^2\theta}}$$

Now as $1-\cos^2\theta = \sin^2\theta$

$$\Rightarrow \int \frac{2\sin^2\theta \cancel{\cos\theta}\, d\theta}{\cancel{\cos\theta}}$$

Now as $\cos 2\theta = 1 - 2\sin^2\theta$

$$\Rightarrow \int (1-\cos 2\theta)\, d\theta$$

$$= \theta - \frac{\sin 2\theta}{2} + c$$

$$= \sin^{-1}\frac{x}{\sqrt{2}} - \frac{x}{\sqrt{2}} \times \sqrt{1-\frac{x^2}{2}} + c$$

Ans $\rightarrow \sin^{-1}\frac{x}{\sqrt{2}} - \frac{x\sqrt{2-x^2}}{2} + c$

12 Problem 8.(a) **8 / 8**

- ✓ **- 0 pts** **Correct**
- **- 2 pts** Algebraic simplification error after the trig sub
- **- 0.5 pts** Minor error (e.g. forgetting a constant or something, incorrect reduction formula)
- **- 1.5 pts** Lack of explanation for integrating sin^2(theta)
- **- 3 pts** Significantly incorrect integration
- **- 2.5 pts** Leaving final answer in terms of theta
- **- 3 pts** Trig sub implemented incorrectly (no differential computed, etc.)

(b) [8pts.] $\int \frac{-2x+1}{x \cdot (x-1)^2} \, dx$.

Now by Partial Fractions

$$\frac{-2x+1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$-2x+1 = A(x-1)^2 + B(x)(x-1) + Cx$$

putting $x=0$

$1 = A \quad \Rightarrow A = 1$

Putting $x=1$

$-1 = C \quad \Rightarrow C = -1$

Putting $x=2$

$-3 = 1(1)^2 + B \times 2 \times 1 - 2$

$-2 = 2B$

$\Rightarrow B = -1$

$$\Rightarrow \int \frac{-2x+1}{x(x-1)^2} dx = \int \frac{1}{x} dx - \int \frac{1}{x-1} dx - \int \frac{1}{(x-1)^2} dx$$

$$= \ln|x| - \ln|x-1| + \frac{1}{(x-1)} + C$$

$$= \ln\left|\frac{x}{x-1}\right| + \frac{1}{x-1} + C$$

13 Problem 8.(b) **8 / 8**

- ✓ **- 0 pts** **Correct**
- **- 1 pts** Forget writing the absolute sign in ln|x| or ln|x+1|
- **- 1 pts** Forget adding constant C to the indefinite integral
- **- 1 pts** Computation mistakes.
- **- 2 pts** More Computation Mistakes
- **- 3 pts** Wrong Factorization at the first place
- **- 4 pts** Omit some important steps in factorization or finding A, B, C.
- **- 6 pts** Find A, B, and C (partial fraction parts) incorrectly or without justification.
- **- 8 pts** Completely Wrong
- **- 4 pts** Incomplete or did not find the indefinite integral to the end.

Problem 9.

(a) [7pts.] Use integration by parts and/or substitution to evaluate the indefinite integral $\int \ln(\sin^2 x) \cdot \cos x \, dx$.

Let $\sin x = t$

$\Rightarrow \cos x \, dx = dt$

$\Rightarrow \int \ln(\sin^2 x) \cos x \, dx = \int \ln(t^2) \, dt$

Now let $f(x) = \ln(t^2)$

$g(x) = 1$

According to Integration by parts

$\int f(x) g(x) \, dx = f(x) \int g(x) \, dx - \int \left(f'(x) \int g(x) \, dx \right) dx$

$\Rightarrow \int \ln(t^2) dt = \ln(t^2) \int dt - \int \frac{2t}{t^2} \times \int dt \, dt$

$\Rightarrow \int \ln(t^2) dt = t \ln(t^2) - \int \frac{2t \times t}{t^2} dt$

$\Rightarrow \int \ln(t^2) dt = t \ln(t^2) - 2t + c$

$= \boxed{\sin x \ln(\sin^2 x) - 2 \sin x + c}$

14 Problem 9.(a) **7 / 7**

✓ **- 0 pts** **Correct**

- 1 pts Missing C

- 1 pts Missing $$I \cdot I$$

- 1 pts Algebra error

- 2 pts Multiple algebra errors

- 4 pts Conceptual problem(s) with substitution or integration by parts

- 7 pts Totally incorrect or not attempted

(b) [7pts.] Note that the function $f(x) = \ln(\sin^2 x) \cdot \cos x$ has an infinite discontinuity at $x = 0$. State if the improper integral $\int_0^{\frac{\pi}{2}} \ln(\sin^2 x) \cdot \cos x \, dx$ converges or not. If it converges, find its value.
Remember to justify your answer.

Now here the discontinuity occurs at lower bound

$$\Rightarrow \int_0^{\pi/2} \ln(\sin^2 x) \cdot \cos x \, dx = \lim_{n \to 0} \int_n^{\pi/2} \ln(\sin^2 x) \cos x \, dx$$

$$\Rightarrow \lim_{n \to 0} \int_n^{\pi/2} \ln(\sin^2 x) \cos x \, dx$$

$$= \lim_{n \to 0} \left(\sin\frac{\pi}{2} \ln\left(\sin^2 \frac{\pi}{2}\right) - 2\sin\frac{\pi}{2} - \sin n \ln(\sin^2 n) + 2\sin n \right)$$

$$= -2 + 2\lim_{n \to 0} \sin n - \lim_{n \to 0} \sin n \cdot \ln(\sin^2 n)$$

$$= -2 - \lim_{n \to 0} \frac{\ln(\sin^2 n)}{1/\sin n}$$

Now as we have an $\frac{-\infty}{\infty}$ form

Using L' Hopital rule

$$= -2 - \lim_{n \to 0} \frac{1}{\cancel{\sin^2 n}} \times \frac{2 \sin n \cancel{\cos n}}{-1 \times \cancel{\cos n}} \times \cancel{\sin^2 n}$$

$$= -2 + 2 \lim_{n \to 0} \sin n = -2$$

as -2 is finite

$\therefore \int_0^{\pi/2} \ln(\sin^2 x)\cos x\,dx$ exists

$\Rightarrow$ THE IMPROPER INTEGRAL CONVERGES

Also its value is -2

15 Problem 9.(b) **7 / 7**

✓ **- 0 pts** **Correct**

- 5 pts The solution does not show the computation of $$\lim_{R\to0^+} (-2\sin R \cdot \ln(\sin R) +2\sin R)$$ (or analogous formula with $$\ln(\sin^2 R)$$)

- 4 pts Did not realise/write that $$\lim_{R\to0^+} \sin R \cdot \ln(\sin R) = 0 \cdot \infty$$ (or analogously with $$\ln(\sin^2 R)$$)

- 3 pts Realised that $$\lim_{R\to0^+} \sin R \cdot \ln(\sin R) = 0 \cdot \infty$$ but did not write that it is an indeterminate form

- 1 pts Forgot $$\cos x$$ in the derivative of $$\frac1{\sin x}$$

- 3 pts Error in part (a) makes problem solvable without L'Hopital's rule

- 1 pts Algebra error

- 7 pts Blank or completely incorrect (e.g., limit at wrong endpoint, does not use the definition of improper integral)

Math 31B
Integration and Infinite Series

Final

Instructions:

- This test is **designed** to take you **3 hours**. On the other hand, per departmental policies, **you are given a 24 hours time window** to work on the exam and submit it: **from Tuesday, March 17th, at 4:30am PDT, to Wednesday, March 18th, at 4:30am PDT**.
- Your submission will be electronic, so please write clearly!
- **Write each part of each question on a dedicated page.** If needed, you can use more than one page.
- Unless you write on the exam itself, **write "Problem x, part (y)" or "Problem x.(y)" at the top of each page** (omit "part (y)" if there is just one part).
- You can use your notes and the textbook (regardless of it being a physical or digital copy). You can also use all past midterms and mock midterms and finals, and all their solutions. You can also use all the lectures on Bruincast.
- You can use a scientific non-graphing calculator.
- Resources other than the ones listed above are not allowed, and will be considered cheating. Collaboration is not allowed and will be considered cheating.
- Unless otherwise stated, you need to justify your answer. Please show all of your work, as partial credit will be given where appropriate, and there may be no credit given for problems where there is no work shown.
- The exam totals 100 points.
- **Submit your exam as a PDF file on Gradescope.** It is your responsibility to make your submission legible. When uploading your work on Gradescope, you have to match each question with the corresponding page.
- If for technological reasons you fail to submit your exam on Gradescope, send it to my email (`marengon@math.ucla.edu`) before the deadline. Note that if your file size is too big to be sent by email, you may have to split it into several pieces.
- In case of late submission, I will deduct 1 point for every minute after the deadline.

FORMULAE

The N-th Taylor polynomial of $f(x)$ at a is given by

$$\sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

and the Taylor series of $f(x)$ at a is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

The Maclaurin series

$$\sin(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$$
$$\cos(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$$

converge absolutely for every $x \in \mathbb{R}$ and the Maclaurin series

$$\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1} \frac{x^n}{n}$$
$$\tan^{-1}(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$$

have radius of convergence 1.

Error bound theorem.
Let $T_N(x)$ be the N^{th} Taylor polynomial centered at $a \in \mathbb{R}$ associated to $f(x)$, and let b in the domain of $f(x)$.
Suppose that $\left|f^{N+1}(u)\right| \leq K$ for all u between a and b. Then

$$|f(b) - T_N(b)| \leq K \, \frac{|b-a|^{N+1}}{(N+1)!}$$

Trigonometric identities

$$\cos^2 x + \sin^2 x = 1$$

$$\begin{aligned}\sin(2x) &= 2\sin(x)\cos(x)\\ \cos(2x) &= \cos^2(x) - \sin^2(x)\\ &= 2\cos^2(x) - 1\\ &= 1 - 2\sin^2(x)\end{aligned}$$

$$\begin{aligned}\sin^2(x) &= \frac{1}{2}(1-\cos(2x))\\ \cos^2(x) &= \frac{1}{2}(1+\cos(2x))\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}(\sin x) &= \cos x\\ \frac{d}{dx}(\cos x) &= -\sin x\\ \frac{d}{dx}(\tan x) &= \sec^2 x\\ \frac{d}{dx}(\sec x) &= \tan x \cdot \sec x\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}\left(\sin^{-1} x\right) &= \frac{1}{\sqrt{1-x^2}} \quad \text{for} \quad |x| < 1\\ \frac{d}{dx}\left(\cos^{-1} x\right) &= -\frac{1}{\sqrt{1-x^2}} \quad \text{for} \quad |x| < 1\\ \frac{d}{dx}\left(\tan^{-1} x\right) &= \frac{1}{1+x^2} \quad \text{for} \quad x \in \mathbb{R}\\ \frac{d}{dx}\left(\sec^{-1} x\right) &= \frac{1}{x\sqrt{x^2-1}} \quad \text{for} \quad x > 1\end{aligned}$$

Hyperbolic identities

$$\begin{aligned}\sinh x &= \frac{e^x - e^{-x}}{2}\\ \cosh x &= \frac{e^x + e^{-x}}{2}\end{aligned}$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\begin{aligned}\frac{d}{dx}(\sinh x) &= \cosh x\\ \frac{d}{dx}(\cosh x) &= \sinh x\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}\left(\sinh^{-1} x\right) &= \frac{1}{\sqrt{x^2+1}} \quad \text{for} \quad x \in \mathbb{R}\\ \frac{d}{dx}\left(\cosh^{-1} x\right) &= \frac{1}{\sqrt{x^2-1}} \quad \text{for} \quad x > 1\end{aligned}$$

Reduction formulas

$$\int \sin^n x \, \mathrm{d}x = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x \, \mathrm{d}x$$

$$\int \cos^n x \, \mathrm{d}x = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x \, \mathrm{d}x$$