1. For each of the following piecewise-defined functions, determine if the function is continuous or not at the specified x value. If the function is not continuous at that point, state the type of discontinuity, and *if possible*, say what can be changed* about the function in order to make it continuous at that point.

   (a) (3 points) Continuous at $x = 3$?

$$f(x) = \begin{cases} x^2 - 5 & \text{if } x < 3 \\ \frac{4}{x-2} & \text{if } x > 3 \\ 5 & \text{if } x = 3 \end{cases}$$

1. $\lim_{x\to 3^-} f(x) = x^2-5 = 3^2-5 = 4$

   $\lim_{x\to 3^+} f(x) = \frac{4}{x-2} = \frac{4}{1} = 4$

   $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$

   $\therefore \lim_{x\to 3} f(x)$ exists ✓

2. $f(3) = 5$

   $\therefore f(3)$ is defined ✓

3. $\lim_{x\to 3} f(x) \neq f(3)$ ✗

$\therefore f(x)$ is not continuous at $x=3$ because $\lim_{x\to 3} f(x) \neq f(3)$. This is a removable discontinuity. To make it continuous, change 5 to 4 for the condition $x=3$, so that $\lim_{x\to 3} f(x) = f(3) = 4$.

   (b) (3 points) Continuous at $x = -4$?

$$g(x) = \begin{cases} \frac{x}{\sqrt{x+5}-1} & \text{if } x \neq -4 \\ 0 & \text{if } x = -4 \end{cases}$$

1. $\lim_{x\to -4} g(x) = \frac{-4}{\sqrt{-4+5}-1} = \frac{-4}{0} = \infty$

   $\lim_{x\to -4^-} g(x) = \frac{\text{negative}}{\text{negative}} = +\infty$

   $\lim_{x\to -4^+} g(x) = \frac{\text{negative}}{\text{positive}} = -\infty$

   $\therefore \lim_{x\to -4} g(x)$ does not exist ✗

2. $g(-4) = 0$

   $\therefore g(-4)$ is defined ✓

3. $\lim_{x\to -4} g(x) \neq g(x)$ ✗

$\therefore g(x)$ is not continuous at $x=-4$ because $\lim_{x\to -4} g(x)$ does not exist. No simple change can be made to make it continuous. This is an infinite discontinuity.

   (c) (3 points) Continuous at $x = 0$?

$$h(x) = \begin{cases} \frac{\sin(x)}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

1. $\lim_{x\to 0^-} h(x) = \frac{\sin(x)}{x} = 1$

   $\lim_{x\to 0^+} h(x) = \frac{\sin(x)}{x} = 1$

   $\lim_{x\to 0^-} h(x) = \lim_{x\to 0^+} h(x)$

   $\therefore \lim_{x\to 0} h(x)$ exists ✓

2. $h(0) = 1$

   $\therefore h(0)$ is defined ✓

3. $\lim_{x\to 0} h(x) = h(0)$ ✓

$\therefore h(x)$ is continuous at $x=0$ because $\lim_{x\to 0} h(x) = h(0)$.

   (d) (3 points) Continuous at $x = 1$?

$$p(x) = \begin{cases} \frac{x^2-1}{x+1} & \text{if } x < 1 \\ x + 1 & \text{if } x \geq 1 \end{cases}$$

1. $\lim_{x\to 1^-} p(x) = \frac{x^2-1}{x+1} = \lim_{x\to 1^-} \frac{(x-1)\cancel{(x+1)}}{\cancel{x+1}} = \lim_{x\to 1^-} (x-1) = 0$

   $\lim_{x\to 1^+} p(x) = \lim_{x\to 1^+} (x+1) = 2$

   $\lim_{x\to 1^-} p(x) \neq \lim_{x\to 1^+} p(x)$

   $\therefore \lim_{x\to 1} p(x)$ does not exist ✗

2. $p(1) = 2$

   $\therefore p(1)$ is defined ✓

3. $\lim_{x\to 1} p(x) \neq p(1)$ ✗

$\therefore p(x)$ is not continuous at $x=1$ because $\lim_{x\to 1} p(x)$ does not exist. To make it continuous, change $x+1$ to $x-1$ for $x \geq 1$. This is a jump discontinuity.

---

*That is, if it's possible to make a very slight change to the function to make it continuous

2. Compute the following limits. If a limit does not exist, be as specific as possible. (E.g. for an infinite limit, find the one-sided limits.) As always, you must justify each answer thoroughly.

   Note: For these limits, and all of the others on this exam, you should compute them using algebraic techniques. Don't just numerically approximate them.

   (a) (6 points) $\lim_{t\to -5}\left(\frac{5t+1}{t^2+t-20}-\frac{3-t}{3t+15}\right)$ plugging in would give 0/0

$$= \lim_{t\to -5}\left(\frac{5t+1}{(t+5)(t-4)}-\frac{3-t}{3(t+5)}\right)$$

$$= \lim_{t\to -5}\left(\frac{3(5t+1)}{3(t+5)(t-4)}-\frac{(3-t)(t-4)}{3(t+5)(t-4)}\right)$$

$$= \lim_{t\to -5}\left(\frac{3(5t+1)-(3-t)(t-4)}{3(t+5)(t-4)}\right)$$

$$= \lim_{t\to -5}\left(\frac{15t+3-(3t-12-t^2+4t)}{3(t+5)(t-4)}\right)$$

$$= \lim_{t\to -5}\left(\frac{15t+3-3t+12+t^2-4t}{3(t+5)(t-4)}\right)$$

$$= \lim_{t\to -5}\left(\frac{t^2+15t-3t-4t+3+12}{3(t+5)(t-4)}\right)$$

$$= \lim_{t\to -5}\left(\frac{t^2+8t+15}{3(t+5)(t-4)}\right)$$

$$= \lim_{t\to -5}\frac{\cancel{(t+5)}(t+3)}{3\cancel{(t+5)}(t-4)}$$

$$= \lim_{t\to -5}\frac{t+3}{3(t-4)}$$

$$= \frac{-5+3}{3(-9)} = \frac{-2}{-27} = \boxed{\frac{2}{27}}$$

   (b) (6 points) $\lim_{x\to 2}\frac{x^2+3x-10}{\sqrt{4x-4}-x}$ plugging in would give 0/0

$$= \lim_{x\to 2}\frac{(x+5)(x-2)}{\sqrt{4x-4}-x}\cdot\frac{\sqrt{4x-4}+x}{\sqrt{4x-4}+x}$$

$$= \lim_{x\to 2}\frac{(x+5)(x-2)(\sqrt{4x-4}+x)}{4x-4-x^2}$$

$$= \lim_{x\to 2}\frac{(x+5)(x-2)(\sqrt{4x-4}+x)}{-(x^2-4x+4)}$$

$$= \lim_{x\to 2}\frac{(x+5)\cancel{(x-2)}(\sqrt{4x-4}+x)}{-\cancel{(x-2)}(x-2)}$$

$$= \lim_{x\to 2}\frac{(x+5)(\sqrt{4x-4}+x)}{-(x-2)}$$

$$= \frac{(2+5)(\sqrt{8-4}+2)}{0}$$

$$= \frac{7(2+2)}{0}$$

$$= \frac{28}{0} = \infty$$

$$\lim_{x\to 2^-}\frac{(x+5)(\sqrt{4x-4}+x)}{-(x-2)} = \frac{\text{positive}}{\text{positive}} = +\infty$$

$$\lim_{x\to 2^+}\frac{(x+5)(\sqrt{4x-4}+x)}{-(x-2)} = \frac{\text{positive}}{\text{negative}} = -\infty$$

3. Define a function f as follows, where a and b are unknown constants:

$$f(x) = \begin{cases} \frac{x}{\sec(x)} \cdot \cot(x) & \text{if } x < 0 \\ ax + b & \text{if } 0 < x \leq 3 \\ \frac{x^2-9}{2x^3-5x^2-9} & \text{if } x > 3 \end{cases}$$

(a) (10 points) Find values of a and b such that both $\lim_{x\to 0} f(x)$ exists and $\lim_{x\to 3} f(x)$ exists.

(b) (2 points) Do the values of a and b that you found in part (a) make f continuous at $x = 0$? If so, why? If not, what kind of discontinuity is this, and can something simple be done to make f continuous at this point? *Be as specific as possible.*

(c) (2 points) Do the values of a and b that you found in part (a) make f continuous at $x = 3$? If so, why? If not, what kind of discontinuity is this, and can something simple be done to make f continuous at this point? *Be as specific as possible.*

a) $\lim_{x\to 0} f(x)$:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{x}{\sec x} \cdot \cot x = \lim_{x\to 0^-} \frac{x\cot x}{\sec x} = \lim_{x\to 0^-} \frac{x\frac{\cos x}{\sin x}}{\frac{1}{\cos x}} = \lim_{x\to 0^-} \frac{x\cos x}{\sin x} \cdot \frac{\cos x}{1} = \lim_{x\to 0^-} \frac{x\cos^2 x}{\sin x} = \lim_{x\to 0^-} \frac{x}{\sin x} \cdot \lim_{x\to 0^-} \cos^2 x = \frac{1}{\lim_{x\to 0^-} \frac{\sin x}{x}} \cdot \lim_{x\to 0^-} \cos^2 x$$

$= \frac{1}{1} \cdot 1^2 = 1$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} ax + b = b$

$\boxed{b=1}$

scratch work:

$$\begin{array}{r} 2x^2+x+3 \\ x-3 \,\big)\, \overline{2x^3-5x^2+0x-9} \\ -\underline{2x^3-6x^2} \\ x^2+0x \\ -\underline{x^2-3x} \\ 3x-9 \\ -\underline{3x-9} \\ 0 \end{array}$$

$\lim_{x\to 3} f(x)$:

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (ax+1) = 3a+1$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \frac{x^2-9}{2x^3-5x^2-9} = \lim_{x\to 3^+} \frac{(x+3)\cancel{(x-3)}}{\cancel{(x-3)}(2x^2+x+3)} = \lim_{x\to 3^+} \frac{x+3}{2x^2+x+3} = \frac{6}{2(9)+3+3} = \frac{6}{24} = \frac{1}{4}$$

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$

$3a+1 = \frac{1}{4}$

$3a = \frac{1}{4} - \frac{4}{4} = \frac{-3}{4}$

$\boxed{a = -\frac{1}{4}}$

b) No, $b=1$ does not make $f(x)$ continuous at $x=0$. The conditions for continuity are 1) $\lim_{x\to 0} f(x)$ exists, 2) $f(0)$ exists, and 3) $\lim_{x\to 0} f(x) = f(0)$. The second condition is not met because $f(x)$ isn't defined at $x=0$ (i.e. $f(0)$ is undefined). This is a removable discontinuity. It can be fixed by changing the condition $0<x\leq 3$ to $0\leq x\leq 3$. That way, $f(0)$ would be defined, and $-\frac{1}{4}(0)+1 = 1$, so $\lim_{x\to 0} f(x)$ would equal $f(0)$. All three conditions would be met.

c) Yes, $a = -\frac{1}{4}$ and $b=1$ make $f(x)$ continuous at $x=3$. The conditions for continuity are 1) $\lim_{x\to 3} f(x)$ exists, 2) $f(3)$ exists, and 3) $\lim_{x\to 3} f(x) = f(3)$. Condition 1 is met since $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$ with the values $a = -\frac{1}{4}$ and $b=1$. Condition 2 is met because $f(3) = ax+b = -\frac{1}{4}(3)+1 = \frac{1}{4}$. Condition 3 is met because $\lim_{x\to 3} f(x) = f(3) = \frac{1}{4}$.

4. For both of the following questions, solve them using the appropriate *limit*. (That is, if you happen to know how to use differentiation rules to solve these problems, you won't get credit for just doing that. You can use that to double-check your answer, though.)

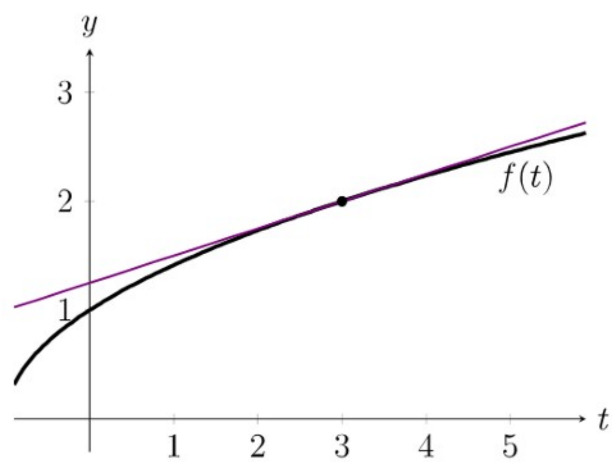
   (a) (6 points) You are studying the pods of orcas in Puget Sound. According to a population model you have come up with, the population of orcas t years from now will be

   $$p(t) = 30 - \frac{4}{t}$$

   How fast will the population be growing 2 years from now? (That is, find the instantaneous rate of change of $p(t)$ at $t = 2$.)

   (b) (6 points) Compute the slope of the tangent line to the graph of $f(t) = \sqrt{t+1}$ at $t = 3$.

   (Note: Don't just approximate the answer! Compute it as a limit. The graph below is for visual aid only.)



a) $p(t) = 30 - \frac{4}{t}$

$\lim\limits_{h\to 0} \frac{f(h+a)-f(a)}{h} = \lim\limits_{h\to 0} \frac{f(h+2)-f(2)}{h}$

$= \lim\limits_{h\to 0} \left(\frac{30 - \frac{4}{h+2} - (30 - \frac{4}{2})}{h} \right) = \lim\limits_{h\to 0} \left(\frac{30 - \frac{4}{h+2} - 30 + 2}{h} \right)$

$= \lim\limits_{h\to 0} \left(\frac{\frac{-4}{h+2} + 2}{h} \right) = \lim\limits_{h\to 0} \frac{\frac{-4}{h+2} + \frac{2(h+2)}{h+2}}{h} = \lim\limits_{h\to 0} \left(\frac{\frac{-4+2(h+2)}{h+2}}{h} \right)$

$= \lim\limits_{h\to 0} \frac{-4+2(h+2)}{h(h+2)} = \lim\limits_{h\to 0} \frac{-4+2h+4}{h(h+2)} = \lim\limits_{h\to 0} \frac{2\cancel{h}}{\cancel{h}(h+2)} = \lim\limits_{h\to 0} \frac{2}{h+2}$

$= \frac{2}{2} = \boxed{1}$

b) $\lim\limits_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim\limits_{h\to 0} \frac{f(3+h)-f(3)}{h} = \lim\limits_{h\to 0} \frac{\sqrt{h+4} - \sqrt{4}}{h}$

$= \lim\limits_{h\to 0} \frac{\sqrt{h+4} - 2}{h} \cdot \frac{\sqrt{h+4}+2}{\sqrt{h+4}+2} = \lim\limits_{h\to 0} \frac{h+4-4}{h(\sqrt{h+4}+2)} = \lim\limits_{h\to 0} \frac{h}{h(\sqrt{h+4}+2)} = \lim\limits_{h\to 0} \frac{1}{\sqrt{h+4}+2}$

$= \frac{1}{\sqrt{0+4}+2} = \frac{1}{2+2} = \frac{1}{4}$

$m = \frac{1}{4}$

$f(3) = \sqrt{3+1} = 2$

$(3, 2)$

$y - 2 = \frac{1}{4}(x-3)$