9

Problem 1.

(a) [2pts.] Which of the following is largest?

A. $\sqrt{5.1} - \sqrt{5}$ B. $\sqrt{8.1} - \sqrt{8}$ C. $\sqrt{12.1} - \sqrt{12}$

$f(x+\Delta x) \approx f(x) + f'(x)\Delta x$

$f(x+\Delta x) - f(x) \approx f'(x)\Delta x$

Let $f(x) = \sqrt{x}$

$f'(x) = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}$

$\therefore f'(x)\Delta x$ is maximum when x is minimum (since $\Delta x = 0.1$ for all options)

$\therefore$ At $x = 5$, $f(x+\Delta x) - f(x)$ $[\sqrt{5.1} - \sqrt{5}]$ is maximum

Ans A

(b) [3pts.] Which of the choices is closest to $\frac{1}{\sqrt{24}}$?

A. 0.2 B. 0.204 C. 0.196 D. 0.3

Let $f(x)$ be defined as $f(x) = \frac{1}{\sqrt{x}}$

$f(25 + (-1)) \approx f(25) + f'(25)\cdot(-1)$ $\quad (\because f(x+\Delta x) \approx f(x) + f'(x)\Delta x)$

$f'(x) = \frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right) = \frac{d}{dx}\left(x^{-1/2}\right) = -\frac{1}{2}x^{-3/2}$

$f'(25) = -\frac{1}{2} \times (25)^{-3/2} = \frac{-1}{250}$

$f(24) \approx f(25) + f'(25)(-1)$

$\Rightarrow f(24) \approx \frac{1}{\sqrt{25}} + \left(-\frac{1}{250} \times (-1)\right)$

$\Rightarrow f(24) \approx \frac{1}{5} + \frac{1}{250}$

$\Rightarrow f(24) \approx 0.2 + 0.004$

$\therefore \frac{1}{\sqrt{24}} \approx 0.204$

Ans B

4

(c) [5pts.] Water pours into a conical tank with a height of H and a radius of R. The water is pouring at a rate of $\frac{dV}{dt} = \rho$. At what rate is the water level rising when the height is h_0? Express your answer in terms of ρ, h_0, H, and R. Note that the volume of a cone with base radius r and height h is $\frac{1}{3}\pi r^2 h$.

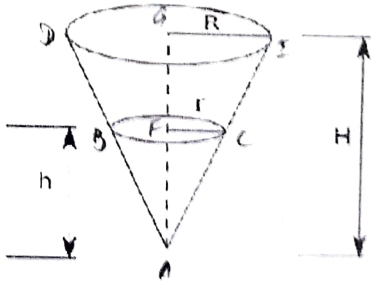


Figure 1

From the figure above, since triangles ΔAFC and ΔAGE are similar

$$\frac{r}{h} = \frac{R}{H}$$

$$\therefore r = \frac{R}{H} \cdot h$$

$$V = \frac{1}{3}\pi r^2 h$$

Substituting $r = \frac{R}{H} \cdot h$

$$V = \frac{1}{3}\pi \left(\frac{R}{H} \cdot h\right)^2 \cdot h$$

$$= \frac{1}{3}\pi \left(\frac{R}{H}\right)^2 h^3$$

$$\frac{dV}{dt} = \frac{d\left(\frac{1}{3}\pi \left(\frac{R}{H}\right)^2 h^3\right)}{dt} = \rho$$

$$\Rightarrow \frac{1}{\cancel{3}}\pi \left(\frac{R}{H}\right)^2 \cdot \cancel{3}h^2 \frac{dh}{dt} = \rho$$

$$\frac{dh}{dt} = \frac{\rho \cdot H^2}{\pi R^2 h^2}$$

At $h = h_0$

$$\frac{dh}{dt} = \frac{\rho H}{\pi R h_0^2}$$

Ans: At height h_0, water level is rising at a rate $= \frac{\rho H}{\pi R h_0^2}$

10

Problem 2.

Consider the function $f(x) = x^5 - 80x$ on the interval $[-1, 3]$.

(a) [2pts.] Find the derivative of f.

$f'(x) = \dfrac{d\,(x^5 - 80x)}{dx} = 5x^4 - 80$

Ans $f'(x) = 5x^4 - 80$

(b) [4pts.] What are the critical points of f in the domain. $[-1, 3]$?

Critical points are points where $f'(x) = 0$ or $f'(x)$ is not defined.

$f'(x) = 5x^4 - 80$

For $f'(x) = 0$,

$5x^4 - 80 = 0$

$\Rightarrow 5(x^4 - 16) = 0$

$\Rightarrow x^4 - 16 = 0$

$\Rightarrow (x^2 + 4)(x^2 - 4) = 0$

$\Rightarrow (x^2 + 4)(x + 2)(x - 2) = 0$

$\therefore f'(x) = 0$ when $x + 2 = 0 \Rightarrow x = -2$ ($\because x^2 + 4 > 0$ for all values of x)

or $x - 2 = 0 \Rightarrow x = 2$

In the domain $[-1, 3]$ the only critical point observed is $x = 2$

Ans Critical point in domain $[-1, 3]$ is present at $x = 2$

(c) [4pts.] Find the absolute minima and maxima (also called the global minima and maxima) of f on $[-1, 3]$ by comparing critical points to endpoints.

In the domain $[-1, 3]$ values of $f(x)$ on critical points and endpoints are

Critical point $(x=2)$:

$f(2) = 2^5 - 80 \times 2$

$= 32 - 160$

$= -128$

End point :

$f(-1) = (-1)^5 - 80 \cdot (-1)$

$= -1 + 80$

$= 79$

$f(3) = 3^5 - 80 \times 3$

$= 243 - 240$

$= 3$

Absolute minima of f on $[-1, 3]$ is at $x = 2$

Absolute maxima of f on $[-1, 3]$ is at $x = -1$

10

Problem 3. *10pts.*

Calculate the derivative of y with respect to x given

$$\sin(x^2y) = (x+y)^3.$$

(You do not need to simplify the expression that you get.)

$$\sin(x^2y) = (x+y)^3$$

Differentiating both sides with respect to x.

$$\frac{d}{dx}(\sin(x^2y)) = \frac{d}{dx}(x+y)^3$$

$$\Rightarrow \cos(x^2y).(2xy + x^2\frac{dy}{dx}) = 3(x+y)^2.(1+\frac{dy}{dx})$$

$$\Rightarrow 2xy\cos(x^2y) + x^2\cos(x^2y)\frac{dy}{dx} = 3(x+y)^2 + 3(x+y)^2\frac{dy}{dx}$$

$$\Rightarrow x^2\cos(x^2y)\frac{dy}{dx} - 3(x+y)^2\frac{dy}{dx} = 3(x+y)^2 - 2xy\cos(x^2y)$$

$$\Rightarrow \frac{dy}{dx}(x^2\cos(x^2y) - 3(x+y)^2) = 3(x+y)^2 - 2xy\cos(x^2y)$$

$$\Rightarrow \frac{dy}{dx} = \frac{3(x+y)^2 - 2xy\cos(x^2y)}{x^2\cos(x^2y) - 3(x+y)^2}$$ [Ans]

Problem 4.

Consider the function

$$f(x) = \frac{2x^3}{(x^2-4x+4)}.$$

(a) [1pts.] What is the domain of f?

Domain of f is all points at which f is defined

f is not defined when $x^2-4x+4=0$.

For $x^2-4x+4=0$

$(x-2)^2=0 \to x=2$

$\therefore$ Domain of f is $(-\infty,2)\cup(2,\infty)$

(b) [4pts.] Find the vertical asymptotic behaviour of f at 2 and describe the horizontal asymptotic behaviour.

$\lim_{x\to2^+} f(x) = \lim_{x\to2^+}\frac{2x^3}{(x-2)^2} = +\infty$

4

$\lim_{x\to2^-} f(x) = \lim_{x\to2^-}\frac{2x^3}{(2-2)^2} = +\infty$

$\therefore$ At $x=2$, $f(x)$ approaches $+\infty$ from both the left and the right sides, thus displaying vertical asymptotic behaviour

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{2x^3}{x^2-4x+4} = \lim_{x\to\infty}\frac{x^3(2)}{x^3(\frac{1}{x}-\frac{4}{x^2}+\frac{4}{x^3})} = \lim_{x\to\infty}\frac{2}{0} = \infty$

$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\frac{2x^3}{(x-2)^2}$ $\because (x-2)^2$ is always greater than 0 and $2x^3$ dominates because of a higher power

Since x^3 is an odd power $\lim_{x\to-\infty}\frac{2x^2}{(x-2)^2} = -\infty$

$\therefore$ The function approaches $+\infty$ when $x\to\infty$ and approaches $-\infty$ when $x\to-\infty$

(c) [3pts.] Find the first derivative of f and the critical points of f.

$$f'(x) = \frac{d}{dx}\left(\frac{2x^3}{x^2-4x+4}\right) = 2\frac{d}{dx}\left(\frac{x^3}{(x-2)^2}\right) = 2\left[\frac{3x^2(x-2)^2 - 2x^3(x-2)}{(x-2)^4}\right]$$

3

$$= 2x^2\cancel{(x-2)}\left[\frac{3(x-2)-2x}{(x-2)^{\cancel{4}\,3}}\right]$$

$$f'(x) = \frac{2x^2[3x-6-2x]}{(x-2)^3} = \frac{2x^3-12x^2}{(x-2)^3}$$

Critical points are points where $f'(x)=0$ or $f'(x)$ is not defined.

$f'(x)$ is not defined at $x=2$

For $f'(x)=0$

$$2x^3-12x^2=0$$
$$x^2(2x-12)=0$$

$x=0$ or $2x-12=0 \Rightarrow x=6$.

$\therefore$ Critical points are $0, 2, 6$

(d) [3pts.] Find the second derivative of f and the points of inflection of f.

$$f''(x) = \frac{d}{dx}\left(\frac{2x^3-12x^2}{(x-2)^3}\right) = 2\frac{d}{dx}\left(\frac{x^3-6x^2}{(x-2)^3}\right)$$

3

$$= 2\left[\frac{(3x^2-12x)(x-2)^3 - 3(x-2)^2(x^3-6x^2)}{(x-2)^6}\right]$$

$$= \frac{2\cancel{(x-2)^2}}{(x-2)^{\cancel{6}\,4}}\left[(3x^2-12x)(x-2) - 3(x^3-6x^2)\right]$$

$$= \frac{2}{(x-2)^4}\left[\cancel{3x^3} - \cancel{6x^2} - \cancel{12x^2} + 24x - \cancel{3x^3} + \cancel{18x^2}\right]$$

$$f''(x) = \frac{48x}{(x-2)^4}$$

Points of inflection are points where $f''(x)=0$ and $f''(x)$ changes sign.

At $x=0$, $f''(x)=0$. For $x>0$ $f''(x)>0$ and for $x<0$ $f''(x)<0$.

$\therefore x=0$ is a point of inflection

$x^2(2x-12)$

(e) [2pts.] Find the signs of the first and second derivatives of f in between the transition points.

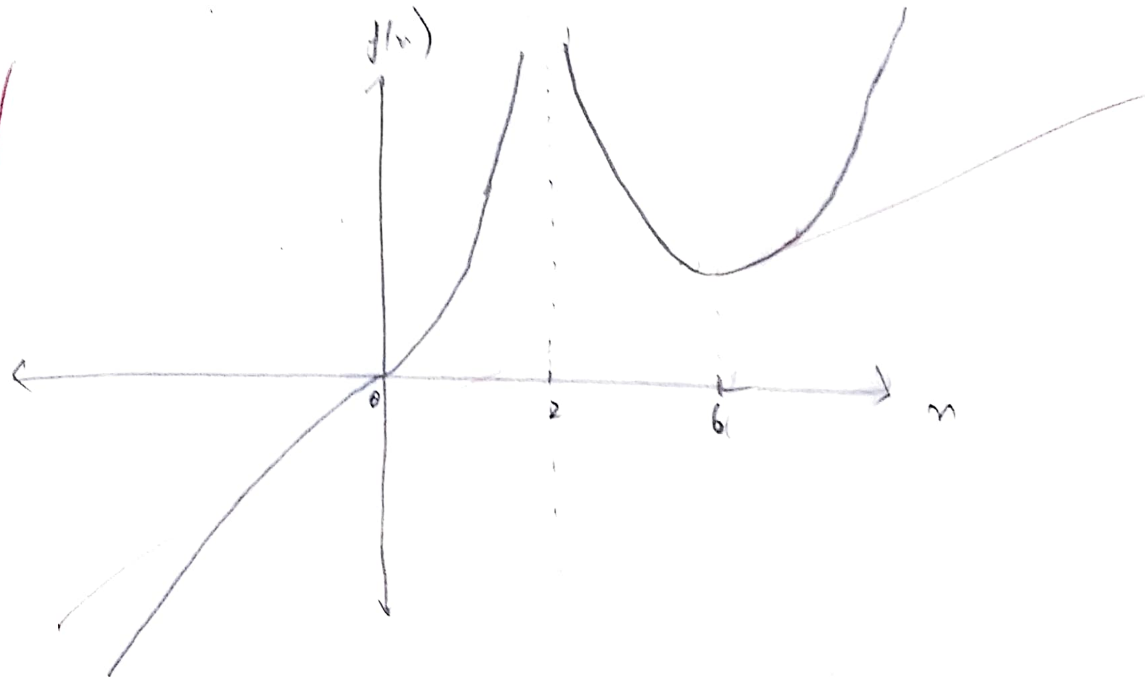
2

Transition points are 0, 2, 6

x	$f'(x)$	$f''(x)$
$(-\infty, 0)$	+	-
$(0,2)$	+	+
$(2,6)$	-	+
$(6,\infty)$	+	+

(f) [5pts.] Draw a graph of f.

4



(g) [2pts.] Observe that as x goes to $+\infty$ and $-\infty$, $f(x)$ is approximated by a line (not a horizontal line, as there is no horizontal asymptote). Find an equation for this line and plot it on your graph above.

1

$x \to +\infty, -\infty$

$f(x)$ is approximated by the line $y = x$