10

**Problem 1.**

Consider the function

$$f(x) = \begin{cases} 2x^3+1 & \text{if } x<1 \\ 3 & \text{if } x=1 \\ \frac{\sqrt{x}-1}{x^2-1} & \text{if } x>1 \end{cases}.$$

(a) [2pts.] Use the limit or continuity laws to determine $\lim_{x\to 1^-} f(x)$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 2x^3+1$ (from function definition - since $f(x) = 2x^3+1$ when $x<1$)

$= 2\times 1^3+1$

$= 2+1$

$= 3$

Ans: 3

(b) [4pts.] Determine $\lim_{x\to 1^+} f(x)$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{\sqrt{x}-1}{x^2-1}$ (from function definition - since $f(x) = \frac{\sqrt{x}-1}{x^2-1}$ when $x>1$)

$\Rightarrow \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{\sqrt{x}-1}{(x^2-1)} \frac{(\sqrt{x}+1)}{(\sqrt{x}+1)}$ [Multiplying by conjugate]

$\Rightarrow \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{\cancel{(x-1)}}{(x+1)\cancel{(x-1)}(\sqrt{x}+1)}$ [$\because x^2-1 = (x+1)(x-1)$]

$\Rightarrow \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{1}{(x+1)(\sqrt{x}+1)}$

$= \frac{1}{(1+1)(\sqrt{1}+1)}$

$= \frac{1}{2\times 2}$

$= \frac{1}{4}$

Ans: $\frac{1}{4}$

(c) [4pts.] Is $f$ left continuous at 1? Is it right continuous at 1? Is it continuous at 1?

~~At~~ $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 2x^3+1$

$= 2\times 1^3+1$

$= 3$

$\cancel{\lim_{x\to 1}} \quad \lim_{x\to 1^+} = \lim_{x\to 1^+} \frac{\sqrt{x}-1}{x^2-1}$

$\Rightarrow \lim_{x\to 1^+} \frac{(\sqrt{x}-1)(\sqrt{x}+1)}{(x^2-1)(\sqrt{x}+1)}$

$\Rightarrow \lim_{x\to 1^+} \frac{\cancel{(x-1)}}{(x+1)\cancel{(x-1)}(\sqrt{x}+1)}$

$\Rightarrow \lim_{x\to 1^+} \frac{1}{(x+1)(\sqrt{x}+1)}$

$= \frac{1}{(1+1)(\sqrt{1}+1)}$

$= \frac{1}{2+2} = \frac{1}{4}$

$f(1) = 3$ (from function definition)

Since $\lim_{x\to 1^-} f(x) = f(1) = 3$, the function is left continuous

Since $\lim_{x\to 1^+} f(x) \neq f(1)$, the function is not right continuous

Since $\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$, the function $f(x)$ is not continuous at 1

**Problem 2.**

Consider the function $f(x) = x^{1/3} + 2 + x^{-1/3}$.

(a) [4pts.] Compute $f'(8)$. (You may use any of the derivative rules we covered in class.)

$f(x) = x^{1/3} + 2 + x^{-1/3}$

$\left.\frac{d\,f(x)}{dx}\right|_{x=8} = \frac{1}{3}x^{-2/3} + 0 - \frac{1}{3}x^{-4/3}\Big|_{x=8}$ [from power rule)

Plugging in $x = 8$:

$f'(8) = \frac{1}{3} \times 8^{-2/3} - \frac{1}{3} 8^{-4/3}$

$= \frac{1}{3} \times \frac{1}{4} - \frac{1}{3} \times \frac{1}{16} = \frac{1}{12} - \frac{1}{48} = \frac{4-1}{48} = \frac{3}{48} = \frac{1}{16}$ (Ans)

(b) [6pts.] Write an equation for a line tangent to $f$ at $x = 8$.

~~At~~ $f(8) = 8^{1/3} + 2 + 8^{-1/3}$

$= 2 + 2 + \frac{1}{2}$

$= 4 + \frac{1}{2}$

$= \frac{9}{2}$

Let $f(x) = y$

Equation of tangent line to $f$ at $x = 8$

$\Rightarrow (y - y_1) = m(x - x_1)$ [slope point form)

Plugging in point $(8, 9/2)$ and slope $m = \frac{1}{16}$

$y - \frac{9}{2} = \frac{1}{16}(x - 8)$

$\Rightarrow 16y - 72 = x - 8$

$\Rightarrow$ ~~16y − x + 6~~ $16y - x - 64 = 0$ (Ans)

**Problem 3.** *10pts.*
Compute the derivative of the function

$$f(x) = \left(1 + \cos^4\left(\sqrt{\sqrt{x^2+3}+1}\right)\right)^8.$$

$$\frac{df(x)}{dx} = 8\left(1+\cos^4\left(\sqrt{\sqrt{x^2+3}+1}\right)\right)^7 \frac{d\left(1+\cos^4\left(\sqrt{\sqrt{x^2+3}+1}\right)\right)}{dx}$$

$$= 8\left(1+\cos^4\sqrt{\sqrt{x^2+3}+1}\right)^7 \cdot 4\cos^3\left(\sqrt{\sqrt{x^2+3}+1}\right) \cdot \left(-\sin\sqrt{\sqrt{x^2+3}+1}\right) \frac{d\left(\sqrt{\sqrt{x^2+3}+1}\right)}{dx}$$

$$\Rightarrow -8\left(1+\cos^4\sqrt{\sqrt{x^2+3}+1}\right)^7 \cdot 4\cos^3\left(\sqrt{\sqrt{x^2+3}+1}\right) \cdot \sin\sqrt{\sqrt{x^2+3}+1} \cdot \frac{1}{2\sqrt{\sqrt{x^2+3}+1}} \cdot \frac{d\left(\sqrt{x^2+3}+1\right)}{dx}$$

$$\Rightarrow -8\left(1+\cos^4\left(\sqrt{\sqrt{x^2+3}+1}\right)\right)^7 \cdot 4\cos^3\sqrt{\sqrt{x^2+3}+1} \cdot \sin\sqrt{\sqrt{x^2+3}+1} \cdot \frac{1}{2\sqrt{\sqrt{x^2+3}+1}} \cdot \frac{1}{\cancel{2}\sqrt{x^2+3}} \cdot \cancel{2}x$$

$$\Rightarrow -\frac{\overset{4}{\cancel{8}}\left(1+\cos^4\left(\sqrt{\sqrt{x^2+3}+1}\right)\right)^7 \cdot 4\cos^3\sqrt{\sqrt{x^2+3}+1} \cdot \sin\sqrt{\sqrt{x^2+3}+1}}{\cancel{2}\sqrt{\sqrt{x^2+3}+1} \cdot \sqrt{x^2+3}}$$

$$\Rightarrow \frac{-16\left(1+\cos^4\sqrt{\sqrt{x^2+3}+1}\right)^7 \cdot \cos^3\sqrt{\sqrt{x^2+3}+1} \cdot \sin\sqrt{\sqrt{x^2+3}+1} \cdot x}{\sqrt{\sqrt{x^2+3}+1} \cdot \sqrt{x^2+3}} \quad (\text{Ans})$$

**Problem 4.**

Define the function $f$ as

$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

4

(a) [4pts.] What is the derivative of $f$ at 1?

$$\left.\frac{d f(x)}{dx}\right|_{x=1} = \cancel{\frac{df}{}} \frac{d\, x^2 \sin\left(\frac{1}{x}\right)}{dx} = 2x \sin\frac{1}{x} + x^2 \cos\frac{1}{x} \cdot \left(-\frac{1}{x^2}\right)\Big|_{x=1}$$

$$= 2 \times 1 \sin(1) + 1 \cos 1 \left(-\frac{1}{1}\right)$$

$$= 2 \sin(1) - \cos 1 \quad (\text{Ans})$$

(b) [6pts.] Is $f$ differentiable at 0? If so, what is the derivative of $f$ at 0? If not, explain why not.

2

~~By squeeze~~ $-1 \leq \sin\frac{1}{x} \leq 1$ for all $x$

~~Hence $\left|\sin\frac{1}{x}\right| \leq |1|$~~

Multiplying both sides by $x^2$

$-x^2 \leq x^2 \sin\frac{1}{x} \leq x^2$ ($\because$ $x^2$ is always positive)

By ~~squeeze theorem~~ $\lim_{x\to 0} -x^2 = \lim_{x\to 0} x^2 = 0$

$\therefore$ By squeeze theorem $\lim_{x\to 0} f(x) = 0$ ($\because$ $f(x)$ is continuous)

$f(0) = \lim_{x\to 0} f(x) = 0$

Hence the function is differentiable at $x = 0$. — does not imply

~~$\left.\frac{df(x)}{dx}\right|_{x=0} = \frac{df}{} \frac{d(x^2 \sin 1/x)}{dx} = 2x \sin 1/x$~~

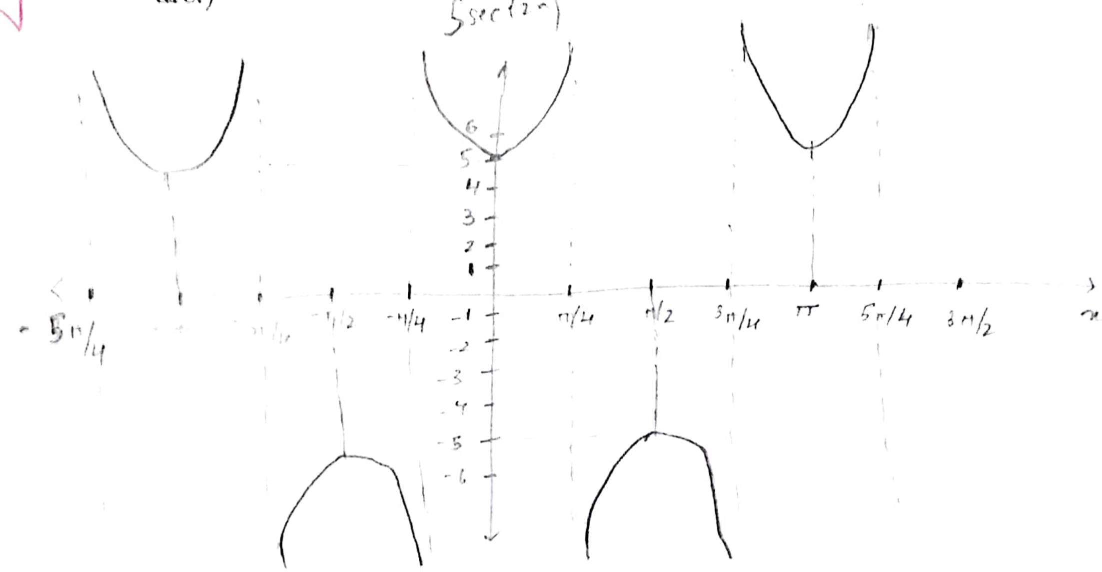
At $x=0$ the $\frac{df(x)}{dx} = \frac{d\, x^2 \sin 1/x}{dx} = 2x \sin 1/x + x^2 \cos\frac{1}{x} \cdot -\frac{1}{x^2}$

$0 + \cos 0 = 1$ (Ans)

sec 0 = 1
sec π/2 = ∞
sec π = -1

**Problem 5.**

(a) [4pts.] Sketch a graph of $5\sec(2x)$. (Your graph just needs to show where the minima are and what values they have, as well as where the vertical asymptotes are.)




(b) [6pts.] Use the intermediate value theorem to show that $5\sec(2x) = \frac{\pi}{2}$ has infinitely many solutions.

Let $n = 1, 2, 3, \ldots$ [~~whole~~ natural numbers]

Consider an interval $[(2n-1)\pi/4, (2n+1)\pi/4]$

Let $f(x)$ be define as $5\sec(2x) - \frac{\pi}{2}$. Hence it is required to prove that $f(x)$ has infinitely many zeroes

$\lim_{x \to (2n-1)\frac{\pi}{4}^+} f(x) = \lim_{x \to (2n-1)\frac{\pi}{4}^+} 5\sec(2x) - \frac{\pi}{2} = +\infty$ ($\because \lim_{x \to (2n-1)\frac{\pi}{4}^+} 5\sec 2x = +\infty$)

$\lim_{x \to n\frac{\pi}{2}} f(x) = \lim_{x \to n\pi/2} 5\sec 2x - \frac{\pi}{2} = \lim_{x \to n\pi/2} 5\sec(n\pi) - \frac{\pi}{2} < 0$ for all $n$

Therefore, there exists a point $p$ s.t. $\in ((2n-1)\frac{\pi}{4}, \frac{n\pi}{2})$ such that $f(x) > 0$

Thus, by the Intermediate value theorem, since $f(p) > 0$ and $f(n\pi/2) < 0$ and $f(x)$ is continuous in interval $[(2n-1)\pi/4, n\frac{\pi}{2}]$, there exists a ~~zero~~ point $c$ such that $f(c) = 0$ in range $((2n-1)\pi/4, \frac{n\pi}{2})$. Thus, there are infinitely many ~~places~~ points where $f(x) = 0 \Rightarrow 5\sec 2x = \frac{\pi}{2}$