Problem 1.

Differentiate the following functions.

(a) [4pts.] $f(x) = \frac{x^2 \sin(x)}{1+x^2}$

(b) [4pts.] $f(x) = \sin^2(3x)\sin(4x^5)$

(c) [4pts.] $f(x) = \sqrt{1+\sqrt{1+\sqrt{1+x}}}$

$$\left(1+\left(1+(1+x)^{1/2}\right)^{1/2}\right)^{1/2}$$

$$\frac{1}{2}\left(1+\left(1+(1+x)^{1/2}\right)^{1/2}\right)^{-1/2} \cdot \left(\frac{1}{2}\left(1+(1+x)^{1/2}\right]^{-1/2}\right)$$

$$\left(\frac{1}{2}(1+x)^{-1/2}\right)$$

$$(1)$$

a) $f(x) = \dfrac{x^2 \sin(x)}{1+x^2}$

$$f'(x) = \frac{(1+x^2)\frac{d}{dx}(x^2\sin(x)) - (x^2\sin(x))\frac{d}{dx}(1+x^2)}{(1+x^2)^2}$$

$$= \frac{(1+x^2)[(2x)(\sin(x)) + x^2\cos(x)] - (x^2\sin(x))(2x)}{(1+x^2)^2}$$

✓4

b) $f'(x) = \frac{d}{dx}[(\sin(3x))^2]\sin(4x^5) + \sin^2(3x)\frac{d}{dx}(\sin(4x^5))$

$$= 2(\sin(3x))(\cos(3x))(3)\sin(4x^5) + \sin^2(3x)(\cos(4x^5))(20x^4)$$

✓4

c) $f'(x) = \dfrac{1}{2\sqrt{1+\sqrt{1+\sqrt{1+x}}}} \cdot \dfrac{1}{2\sqrt{1+\sqrt{1+x}}} \cdot \dfrac{1}{2\sqrt{1+x}}$

✓4

Problem 2.

Suppose $f(x)$ and $g(x)$ are functions and you are told the following information.

- The tangent line to $y = g(x)$ at $x = 1$ is $y = 2x + 1$.
- The tangent line to $y = f(x)$ at $x = 1$ is $y = 4x + 6$.
- The tangent line to $y = f(x)$ at $x = 3$ is $y = 5x - 3$.

(a) [3pts.] What does information does the first bullet point give you?

(b) [~~7pts~~.] What is the tangent line to $y = h(x) = 4x^2 + f(g(x))$ at $x = 1$?

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a) This tells me that the slope of $g(x)$ is equal to 2

b)

$h'(x) = 8x + f'(g(x))\,g'(x)$

$h'(1) = 8(1) + f'(g(1))\,g'(1)$

$= 8 + f'(3)\,g'(1)$

$= 8 + 12(2)$

$= 32$

$\boxed{y - 10 = 32(x-1)}$

$g(x) = 2x+1$ $\quad$ $g(1) = 2(1) + 1 = 3$

$g'(x) = 2$

$f(x) = 4x+6$ $\quad$ $f(1) = 4(1)+6 = 10$

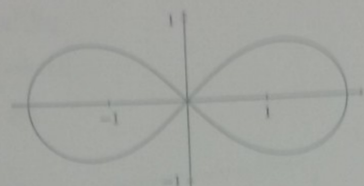
$f'(x) = 4$

$f'(3) = 5(3) - 3 = 12$

Problem 3.

Consider the curve described by the equation

$$(x^2+y^2)^2 = 4(x^2-y^2).$$



(a) [6pts.] Use the method of implicit differentiation to give a formula for y' in terms of x and y.
[After you differentiate, it might help your algebra to temporarily let $r^2=(x^2+y^2)$.]

(b) [4pts.] Find the four points where the tangent line to the curve is horizontal.
[None of them have x-coordinate equal to 0; the only point on the curve whose x-coordinate is equal to 0 is $(0,0)$; your formula for y' is undefined here.]

$(2x^2+2y^2)$

a) $2(x^2+y^2)(2x+2yy') = 4(2x-2yy')$

$\cancel{2}x^3+4x^2yy'+4xy^2+4y^3y' = 8x-8yy'$

$4x^2yy'+4y^3y'+8yy' = 8x-2x^3-4xy^2$

$y'(4x^2y+4y^3+8y) = 8x-2x^3-4xy^2$ (−1)

$$\boxed{y' = \frac{8x-\cancel{2x^3}-4xy^2}{4x^2y+4y^3+8y}}$$

b) $0 = 8x-2x^3-4xy^2$

$4xy^2 = 8x-2x^3$

$y^2 = \frac{8x-2x^3}{4x} = \frac{8x}{4x} - \frac{2x^3}{4x} = 2-\frac{x^2}{2}$

$y^2 = 2-\frac{x^2}{2}$ $\rightarrow (x^2+(2-\frac{x^2}{2})) = 4(x^2-(2-\frac{x^2}{2}))$

$\frac{x^2}{2}+y^2 = 2$ $\qquad \frac{x^2}{2}+2 = 4(\frac{3x^2}{2}-2)$

(−2)

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Problem 4.

Calculate the following limits.

(a) [5pts.] $\lim_{x\to 4}\left[(x^2-16)\frac{x-4}{|x-4|}\right]$ $\left\{ \begin{array}{ll} x>4 & (x-4) \\ x<4 & -(x-4) \end{array} \right.$

(b) [5pts.] $\lim_{x\to 4}\frac{x-4}{\sqrt{x}-\sqrt{8-x}}$

(c) [5pts.] $\lim_{x\to 0}\left[x\sin(\frac{1}{x})\right]$

(d) [5pts.] $\lim_{x\to 3^-}\frac{x-4}{x^2-9}$

a) $\lim_{x\to 4^-} (x^2-16)\frac{\cancel{(x-4)}}{\cancel{(x-4)}} = 0$ $\boxed{\lim_{x\to 4} f(x) = 0}$ 5

$\lim_{x\to 4^+} (x^2-16)\frac{\cancel{(x-4)}}{-\cancel{(x-4)}} = 0$

b) $\lim_{x\to 4} \frac{x-4}{\sqrt{x}-\sqrt{8-x}}\cdot\frac{\sqrt{x}+\sqrt{8-x}}{\sqrt{x}+\sqrt{8-x}} = \frac{\cancel{(x-4)}(\sqrt{x}+\sqrt{8-x})}{8} = \frac{0}{8} = \boxed{0}$ 2

c) $\lim_{x\to 0}\left[x\sin(\frac{1}{x})\right] = \quad -1 \le \sin(\frac{1}{x}) \le 1$ 3

$-x \le x\sin(\frac{1}{x}) \le x$

$\lim_{x\to 0} -x = 0 \qquad \lim_{x\to 0} x = 0$

Therefore,

$\boxed{\lim_{x\to 0} x\sin(\frac{1}{x}) = 0}$

d) $\lim_{x\to 3^-} \frac{x-4}{x^2-9} = \frac{-1}{0} = \pm\infty$

$\frac{2-4}{2^2-9} = \frac{2}{5}$ 3

$\frac{1-4}{1^2-9} = \frac{3}{8}$

$\frac{0-4}{0-9} = \frac{4}{9}$

Problem 5. *2pts.*

This is a bonus question, which is why there are so few points attached to it.

Recall the definition of the derivative of a function $f(x)$:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Let

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{when } x \neq 0, \\ 0 & \text{when } x = 0. \end{cases}$$

What is $f'(0)$?

$$f'(0) = \frac{f(h) - f(0)}{h}$$