MATH 180 GRAPH THEORY: FINAL

• The final is due on Gradescope on *Wednesday, June 10 at 8am (Pacific Time).* No late submissions will be accepted.
• Each problem is worth the same number of points.
• Use of textbooks and internet is allowed. For each problem, all sources must be clearly listed.
• Collaboration is *not* allowed. Asking other people (e.g. on the internet) for help is *not* allowed. Cheating will be reported.
• Please submit good quality scans of your work! (e.g. google "phone scan app")
• Justify all your answers by rigorous proofs.
• $\boxed{\text{Box}}$ all answers.
• Unless stated otherwise, all graphs are assumed to be simple and undirected.

Problem 1. For a graph $G = (V, E)$, let $\overline{G} := \left(V, \binom{V}{2} \setminus E\right)$ denote its complement. Decide whether each of the following statements is true or false.
(a) If G' is a subgraph of G then $\overline{G'}$ is a subgraph of $\overline{G}$.
(b) If G' is a subgraph of G then $\overline{G}$ is a subgraph of $\overline{G'}$.
(c) If G' is an induced subgraph of G then $\overline{G'}$ is an induced subgraph of $\overline{G}$.
(d) If G' is an induced subgraph of G then $\overline{G}$ is an induced subgraph of $\overline{G'}$.

Problem 2. Let $G = (V, E)$ be a graph with $V = \{v_1, v_2, \dots, v_n\}$. Let A_G be its adjacency matrix. Consider the matrix $B = (b_{i,j})_{i,j=1}^n$ given by $B := (A_G)^2$.
(i) Find the trace of B.
(ii) Let $G' = (V, E')$ be the graph whose edges correspond to nonzero off-diagonal entries of B:
$$E' := \{\{v_i, v_j\} \mid i \neq j \text{ such that } b_{i,j} \neq 0\}.$$
For which graphs G will G' be connected?

Problem 3. In each case, decide whether there exists a graph with a given score sequence:
(a) $(9, 8, 7, 6, 5, 4, 3, 2, 1)$;
(b) $(5, 5, 4, 4, 4, 3, 2, 2, 1)$;
(c) $(2, 1, 1, 1, 1, 1, 1, 0, 0)$;
(d) $(4, 4, 4, 4, 4, 4, 4, 4)$;
(e) $(4, 4, 4, 4, 2)$.

Problem 4. For each $m, n \geq 1$,
(i) find the number of Hamiltonian cycles[1] in K_n and $K_{m,n}$;
(ii) decide whether K_n and $K_{m,n}$ is Eulerian.

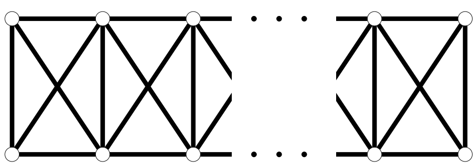
Problem 5. Let $G = (V, E)$ be a connected graph, and let $H = (V, E')$ be given by
$$E' := E \cup \{\{v, w\} \mid v, w \in V \text{ such that } d_G(v, w) = 2\}.$$
Show that H is 2-connected.

[1] Two Hamiltonian cycles are considered the same if they use the same set of edges.

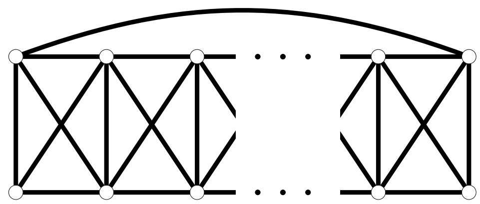
Problem 6. Find the number of pairwise non-isomorphic trees $T = (V, E)$ such that:
(a) $|V| = 3$;
(b) $|V| = 4$;
(c) $|E| = 6$ and T has exactly 3 leaves;
(d) $|E| = 6$ and T has exactly 4 leaves.
In each case, draw these trees.

Problem 7. For which $m, n \geq 1$ does $K_{m,n}$ have a spanning tree whose complement[2] inside $K_{m,n}$ is also a spanning tree of $K_{m,n}$?

Problem 8. For each $n \geq 3$, let G_n and G'_n be the graphs with $2n$ vertices shown below.
(i) For which n is G_n planar?
(ii) For which n is G'_n planar?



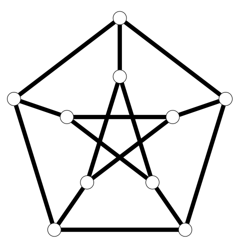
G_n



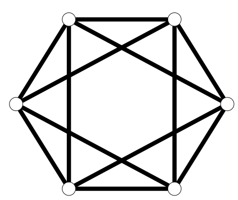
G'_n

Problem 9. Which of the following statements are true for any planar graph G with at least 3 vertices?
(a) G has at least 1 vertex of degree at most 5.
(b) G has at least 2 vertices of degree at most 5.
(c) G has at least 3 vertices of degree at most 5.

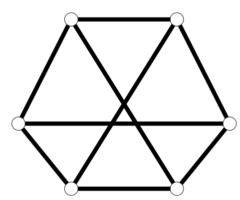
Problem 10. Find the chromatic number of each of the following graphs.



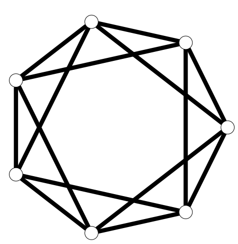
(a)



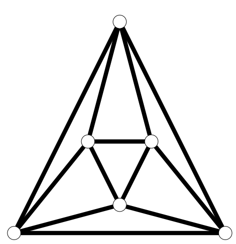
(b)



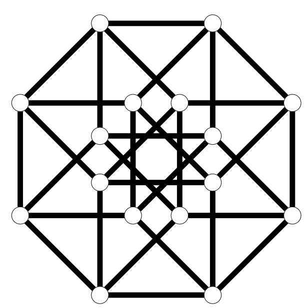
(c)



(d)



(e)



(f)

[2]By a *complement* of a spanning tree T *inside* $K_{m,n}$ we mean the graph $(V(K_{m,n}), E(K_{m,n}) \setminus E(T))$.

20S-MATH180-1 Final

[redacted]

TOTAL POINTS

196.5 / 200

QUESTION 1

1 Problem 1 20 / 20

✓ **- 0 pts** **full credit**

- 4 pts Incorrect answer for part [a], incorrect justification

- 4 pts Incorrect answer for part [c], incorrect justification

- 5 pts Incorrect answer for part [d], no justification

QUESTION 2

2 Problem 2 18 / 20

- 0 pts Full credit [5pts for i, 15pts for ii]

- 1 pts It's not necessary for G to have an even-length path between any two points, just an even-length walk. \[Consider K_3 with an extra leaf attached].

When you turn a path in G' to a walk in G, there may be repetition among the vertices.

- 1 pts Your expression for the trace, while true, doesn't really say what it is in terms of the graph (i.e. it's twice the number of edges aka the sum of the vertex degrees).

✓ **- 2 pts** **You've forgotten the condition that G also needs to be connected.**

- 8 pts For part [ii]; if the graph is bipartite, then G' will not be connected, since you can't escape the bipartite pieces.

- 15 pts No credit for part [ii]

- 8 pts For part [ii], yes, G' will be connected if the graph is complete, but there's a lot more cases where G' is connected.

- 5 pts For part \[ii], yes, all triangles \[aka every two points having a path of length 2] will work, but there are other cases, such as a 5-cycle.

We really want all pairs of points to have a walk of even length.

- 7 pts For part \[ii], yes, trees do not have this property. But there are other graphs (such as K_{3,3}) for which this doesn't work.

- 8 pts Yes, we need the graph to have no isolated vertices [or pairs of vertices], but the real condition is much stronger.

1 also need G to be connected

QUESTION 3

3 Problem 3 20 / 20

✓ **- 0 pts** **All parts correct and justified**

- 2 pts Error in part [e] explanation, but correct answer

QUESTION 4

4 Problem 4 18.5 / 20

- 0 pts Full credit

- 1.5 pts Forgot to divide by 2 in either part of [i]; we could traverse a Hamiltonian cycle in either one of two directions.

✓ **- 1 pts** **Forgot small cases; either n = 1, 2 for K_\[n] or n = m = 1 for K_{n, m}**

✓ **- 0.5 pts** **Forgot small cases for both parts of [i]**

- 3 pts In the second half of part [i]; if n is not equal to m, we get no Hamiltonian cycles in this way.

- 10 pts missing part [ii] - Eulerianness

- 2 pts Error in part [ii]

2 this is not true for n = 1

3 this is not true for n = 1, 2

QUESTION 5

5 Problem 5 20 / 20

✓ **- 0 pts** **Full credit**

- 5 pts Proof has correct ideas, but a large gap.

- 6 pts Proves - correctly - that all vertices are contained in a cycle in H. But this does not imply 2-connectedness.

- 10 pts incorrect proof

QUESTION 6

6 Problem 6 20 / 20

✓ **- 0 pts** **Correct**

- 3 pts (c) wrong answer: the correct answer is 3

- 2 pts (c) justification missing

- 3 pts (d) wrong answer: the correct answer is 4

- 2 pts (d) justification missing

- 1 pts (c) insufficient justification

- 1 pts (d) insufficient justification

QUESTION 7

7 Problem 7 20 / 20

✓ **- 0 pts** **Correct**

- 5 pts wrong answer: the correct answer is m=4, n=3 or m=3, n=4

- 5 pts incorrect proof

- 3 pts Did not actually provide an example of such a tree

- 3 pts a *spanning* treee must use all vertices

QUESTION 8

8 Problem 8 20 / 20

✓ **- 0 pts** **Correct**

- 5 pts (i) wrong answer [Correct answer is "for all n"]

- 5 pts (ii) wrong answer [Correct answer is "only for n=3"]

- 5 pts (ii) Did not prove non-planarity for n>3

- 3 pts (ii) insufficient justification

- 2 pts (ii) insufficient justification

- 5 pts (i) incorrect proof

- 5 pts (ii) incorrect proof

QUESTION 9

9 Problem 9 20 / 20

✓ **- 0 pts** **Correct**

- 5 pts (c) wrong answer [Correct answer: True]

- 8 pts (b,c) wrong answer [Correct answer: True for both]

- 5 pts (c) incorrect proof

- 12 pts (b,c) missing

QUESTION 10

10 Problem 10 20 / 20

✓ **- 0 pts** **Correct**

- 7 pts justification missing: why are these numbers minimal possible?

- 3 pts colorings not shown

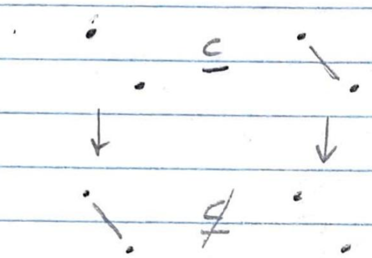
- 3 pts (a) wrong answer [Correct answer: 3]

[redacted]

Math 180 Final Exam

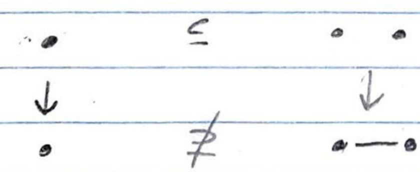
1a. $\boxed{\text{False}}$

Pf. • Consider $G' = (\{1,2\}, \emptyset)$ and $G = (\{1,2\}, \{\{1,2\}\})$
• Note $G' \subseteq G$ (we use this as our subgraph notation)
but $\overline{G'} = G \nsubseteq G' = \overline{G}$



1b. $\boxed{\text{False}}$

Pf. • Consider $G' = (\{1\}, \emptyset)$ and $G = (\{1,2\}, \emptyset)$
• Note $G' \subseteq G$.
• but $\overline{G} = (\{1,2\}, \{\{1,2\}\}) \nsubseteq G' = \overline{G'}$.



1c. $\boxed{\text{True}}$

Pf. · (We use $\subseteq_{ind}$ to indicate induced subgraph)

· Let $G' = (V', E')$, $G = (V, E)$

· Suppose $G' \subseteq_{ind} G$

· Then $V' \subseteq V$ and $E' = \binom{V'}{2} \cap E$

· Then

$$\begin{aligned}
\binom{V'}{2} \cap (E(\overline{G})) &= \binom{V'}{2} \cap \left(\binom{V}{2} \setminus E\right) \\
&= \binom{V'}{2} \setminus E \qquad \left(\text{since } \binom{V'}{2} \subseteq \binom{V}{2}, \text{ since } V' \subseteq V\right) \\
&= \binom{V'}{2} \setminus E' \qquad \left(\text{since } E' = \binom{V'}{2} \cap E\right) \\
&= E(\overline{G'})
\end{aligned}$$

· Also $V(\overline{G'}) = V' \subseteq V = V(\overline{G})$

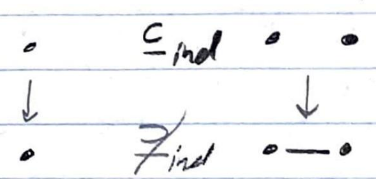
· Thus $\overline{G'} \subseteq_{ind} \overline{G}$ by definition

1d. $\boxed{\text{False}}$

Pf. · Let G', G be defined as in 1b.

· Note $G' \subseteq_{ind} G$

· but $\overline{G} \not\subseteq \overline{G'}$, so it follows $\overline{G} \not\subseteq_{ind} \overline{G'}$



1 Problem 1 **20 / 20**

✓ **- 0 pts** **full credit**

- 4 pts Incorrect answer for part [a], incorrect justification

- 4 pts Incorrect answer for part [c], incorrect justification

- 5 pts Incorrect answer for part [d], no justification

2i. $\cdot \operatorname{tr}(B) = \sum_{i=1}^{n} b_{ii}$

$= \boxed{\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} a_{ji}}$

$= \sum_{i=1}^{n} \#\{\text{Closed walks length-2 from } v_i \text{ to } v_i \text{ in } G\}$

$= \boxed{\#\{\text{Closed walks length-2 in } G\}}$

(here we consider each walk as "ordered" ie with a start and end.)

- For any sets A, B, define $E(A,B) = \{\{u,v\} : u \in A, v \in B\}$

- Let G, G', defined as in problem statement

2ii. <u>Claim:</u> G' connected $\Leftrightarrow$ G not bipartite

- <u>Pf:</u> - ($\Rightarrow$) - Suppose G is bipartite

  - Then $V = A \dot{\cup} B$, where $E \subseteq E(A,B)$ (for some $A, B \subseteq V$)

  - Let $u \in A$, $v \in B$

  - Then $u = v_i$, $v = v_j$ for some $i,j \in [n]$ (note $i \neq j$)

  - Now $b_{ij} = a_{ij}^{(2)} = \#\{$walks from v_i to v_j in G length-2$\}$
    (from lecture)

  - But $\nexists$ walk from v_i to v_j in G length-2
    (if there were such a walk $W = (v_i, v', v_j)$,
    then $\{v_i, v'\}, \{v', v_j\} \in E \Rightarrow v' \in B$ and $v' \in A$, $\lightning$)

  - Thus $b_{ij} = 0$

  - Therefore $\{u, v\} = \{v_i, v_j\} \notin E'$ (by def.)

  - Thus A and B are disjoint vertex sets in G'

  - Thus G' is disconnected

($\Leftarrow$) · Suppose G' disconnected

· Then $\exists A, B \subseteq V$ st. $V = A \dot{\cup} B$ and $E' \cap E(A,B) = \emptyset$.

· <u>Subclaim 1:</u> · $\forall u \in A$ $\forall v \in B$ $\nexists$ walk from u to v in G, length 2.

<u>Pf:</u> · Note $u = v_i$ and $v = v_j$ for some $i, j \in [n]$ (note $i \neq j$)

·

· Since $\{u,v\} \notin E'$ (since $\{u,v\} \in E(A,B)$ and $E(A,B) \cap E' = \emptyset$)
and $i \neq j$, $b_{ij} = 0$ (by def of E')

· Thus $0 = b_{ij} = a_{ij}^{(2)} = \#\{\text{walks from } v_i \text{ to } v_j \text{ in } G, \text{length } 2\}$
(result from lecture) $\square$

· <u>Subclaim 2:</u> · $\forall X, Y \subseteq V$ st. $X \dot{\cup} Y = V$, $E \cap E(A,B) \neq \emptyset$

<u>Pf:</u> · Let $u \in X$, $v \in Y$.
· Since G connected, $\exists$ path P from u to v in G
· This path must cross from X to Y at some point (possibly multiple times) clearly, so $\exists e \in E$ st. $e \in E(A,B)$. $\square$

- Let $A_0 = A$, $B_0 = B$
- Pick any $e = \{u_0, w_0\} \in E$ st. $u_0 \in A_0$ and $w_0 \in B_0$
(exists by subclaim 2)

- If $\{w_0, v'\} \in E$ for some $v' \in B_0$, then $W = (u_0, w_0, v')$ is a walk from u_0 to w_0 in G length 2, contradiction of subclaim 1 $\lightning$

Thus $\{w_0, v'\} \notin E$ $\forall v' \in B_0$

- Let $A_1 = A_0 \cup \{w_0\}$ and $B_1 = B_0 \setminus \{w_0\}$

- Pick any $e_1 = \{u_1, w_1\}$ st. $u_1 \in A_1$ and $w_1 \in B_1$. Note $u_1 \in A$.
(exists by Subclaim 2)

- If $\{w_1, v'\} \in E$ for some $v' \in B_1$, then $W = (u_1, w_1, v')$ is a walk from u_1 to w_1 in G length 2 contradiction of subclaim 1 $\lightning$

Thus $\{w_1, v'\} \notin E$ $\forall v' \in B_0$

- Let $A_2 = A_1 \cup \{w_1\}$ and $B_2 = B_1 \setminus \{w_1\}$

$\vdots$

- Continuing in this manner (this is really just induction) we will eventually exhaust B, that is, eventually $B_k = 0$.

- Thus $B = \{w_0, w_1, \dots w_{k-1}\}$

- By the above logic, this is an empty set in G $(E \cap \binom{B}{2} = \emptyset)$

- Identical logic shows A is a empty set in G $(E \cap \binom{A}{2} = \emptyset)$

- Thus $G = A \dot{\cup} B$ is bipartite (since $E \subseteq E(A,B)$) $\square$

2 Problem 2 **18 / 20**

- **- 0 pts** Full credit [5pts for i, 15pts for ii]
- **- 1 pts** It's not necessary for G to have an even-length path between any two points, just an even-length walk. \[Consider K_3 with an extra leaf attached].

  When you turn a path in G' to a walk in G, there may be repetition among the vertices.
- **- 1 pts** Your expression for the trace, while true, doesn't really say what it is in terms of the graph (i.e. it's twice the number of edges aka the sum of the vertex degrees).
- ✓ **- 2 pts You've forgotten the condition that G also needs to be connected.**
- **- 8 pts** For part [ii]; if the graph is bipartite, then G' will not be connected, since you can't escape the bipartite pieces.
- **- 15 pts** No credit for part [ii]
- **- 8 pts** For part [ii], yes, G' will be connected if the graph is complete, but there's a lot more cases where G' is connected.
- **- 5 pts** For part \[ii], yes, all triangles \[aka every two points having a path of length 2] will work, but there are other cases, such as a 5-cycle.

  We really want all pairs of points to have a walk of even length.
- **- 7 pts** For part \[ii], yes, trees do not have this property. But there are other graphs (such as K_{3,3}) for which this doesn't work.
- **- 8 pts** Yes, we need the graph to have no isolated vertices [or pairs of vertices], but the real condition is much stronger.

1 also need G to be connected

· Recall from lecture

3. <u>Useful Fact.</u> $D = (d_1, d_2, \dots d_n)$ is a graph score

$\iff D' = (d_2 - 1, d_3 - 1, \dots d_{d_1+1} - 1, d_{d_1+2}, \dots d_n)$ is a graph score

(where D is in nonincreasing order)

3a. · $\boxed{\text{No}}$ ∴ Recall that for any (simple) graph G,

$\forall v \in V(G)\ \deg_G(v) \leq |V(G)| - 1$

· But if some G had score$(G) = (9, 8, 7, \dots, 1)$

then $\deg_G(v_1) = 9 > 8 = |V(G)| - 1$ ↯

3b. · $\boxed{\text{Yes}}$ $(5\ 5\ 4\ 4\ 4\ 3\ 2\ 2\ 1)$ (✯)

$\to (4\ 3\ 3\ 3\ 2\ 2\ 2\ 1)$

$\to (2\ 2\ 2\ 1\ 2\ 2\ 1)$

$(2\ 2\ 2\ 2\ 2\ 1\ 1)$

$\to (1\ 1\ 2\ 2\ 1\ 1)$

$(2\ 2\ 1\ 1\ 1\ 1)$

$\to (1\ 0\ 1\ 1\ 1)$

$(1\ 1\ 1\ 1\ 0)$

$\to (0\ 1\ 1\ 0)$

$(1\ 1\ 0\ 0)$

$\to (0\ 0\ 0)$

· The empty graph on 3 vertices has score$(E_3) = (0\ 0\ 0)$,

so $\exists$ graph G with score$(G) = $ (✯) (by Useful Fact)

3c. $\boxed{\text{Yes}}$ • Consider G drawn below:

• Note $\text{score}(G) = (2\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 0)$

3d. $\boxed{\text{Yes}}$ • $(4\ 4\ 4\ 4\ 4\ 4\ 4\ 4)\ (*)$

$\to (3\ 3\ 3\ 3\ 4\ 4\ 4)$

$(4\ 4\ 4\ 3\ 3\ 3\ 3)$

$\to (3\ 3\ 2\ 2\ 3\ 3)$

$(3\ 3\ 3\ 3\ 2\ 2)$

$\to (2\ 2\ 2\ 2\ 2)$

$\to (1\ 1\ 2\ 2)$

$(2\ 2\ 1\ 1)$

$\to (1\ 0\ 1)$

$(1\ 1\ 0)$

$\to (0\ 0)$

• Note $\text{score}(E_2) = (0\ 0)$ (Empty graph on 2 vertices)
so by useful Fact $\exists$ graph G w $\text{score}(G) = (*)$

3e · $\boxed{\text{No}}$

· $(4\ 4\ 4\ 4\ 2)\ (\Diamond)$

$\to (3\ 3\ 3\ 1)$

$\to (2\ 2\ 0)$

$\to (1\ -1)$

· clearly $\nexists$ graph G with score $(G) = (1, -1)$

· Thus by Useful Fact, $\nexists$ graph G with score $(G) = (\Diamond)$

3 Problem 3 20 / 20

✓ **- 0 pts** **All parts correct and justified**

- 2 pts Error in part [e] explanation, but correct answer

4i. • First we count the number of oriented Hamiltonian cycles

• We define an <u>oriented hamiltonian cycle</u> of G as a Ham. cycle that has been given a specified path to it, ie a distinct list of vertices $(v_1, v_2, \dots v_n)$ st. $\{v_i, v_{i+1}\} \in E(G)$ $\forall i \in [n-1]$

• Define $OHam(G)$ as the set of all oriented Hamiltonian cycles in G
• Define $UOHam(G)$ as the set of all unordered Hamiltonian cycles in G.
•
• Define $Perm(S)$ as the set of permutations of S for any set S.

•

•

• Let $n \in \mathbb{N}$.
• Note $OHam(K_n) = Perm(V(K_n))$ (since $\{u,v\} \in E(K_n)$ $\forall v,u \in V(K_n)$)

• Thus $|OHam(K_n)| = |Perm(V(K_n))| = n!$

• We now fix the overcounting.

• Define the equivalence relation on $OHam(G)$ $\sim$ by $H_1 \sim H_2 \iff H_1$ can be reoriented to yield H_2 (that is H_1 can be rotated and/or reversed to yield H_2) $\forall H_1, H_2 \in OHam(G)$

• Note $UOHam(G)$ is in bijection with the equivalence classes of $OHam(G)$, that is $|UOHam(G)| = |OHam(G)/\sim|$

- Lastly, note each $H \in OHam(G)$ has exactly $2|V(G)|$ distinct reorientations ($|V(G)|$ rotations, each with a unique reflection)

- That is, $|OHam(G)/H| = 2|V(G)| \quad \forall H \in OHam(G)$

- Thus $|UOHam(G)| = \dfrac{|OHam(G)|}{2|V(G)|}$ (★)

- By (★), $|UOHam(K_n)| = \boxed{\dfrac{n!}{2n}}$

- Let $n, m \in \mathbb{N}$
- Note that if $n \neq m$, $|OHam(K_{n,m})| = 0$
  (this is clear from the fact that any attempt at a Hamiltonian cycle will eventually "run out" of vertices on the smaller part of $K_{n,m}$)
  (more precisely, this is because no sequence of elements from disjoint A, B $|A| \neq |B|$ can alternate between A and B, and hit each element in A and B exactly once, and return to where it started)

- Define $Perm(S, T)$ as the set of "alternating permutations" between S & T, that is $Perm(S,T) = \{(a_1, b_1, a_2, b_2, a_3, b_3, \dots a_n, b_n) :$
  $(a_1, \dots a_n) \in Perm(S)$ and $(b_1, b_2, \dots b_n) \in Perm(T)\}$
  (note $|S| = |T|$ is a requirement for $Perm(S,T) \neq 0$)

- Suppose $n = m$

- Let $A = \{v_1, v_2, \dots v_n\}$ and $B = \{u_1, u_2, \dots u_m\}$
(so A, B are the two parts of $K_{n,m}$)

- Note $OHam(K_{n,m}) = Perm(A,B) \cup Perm(B,A)$
(since $\{u,v\} \in E(K_{n,m})$ $\forall u \in A, \forall v \in B$)

- Thus
$$\begin{aligned}
|OHam(K_{n,m})| &= |Perm(A,B) \cup Perm(B,A)| \\
&= |Perm(A,B)| + |Perm(B,A)| \\
&= |Perm(A)||Perm(B)| + |Perm(B)||Perm(A)| \\
&= 2|Perm(A)||Perm(B)| \\
&= 2(n!)^2
\end{aligned}$$

- Then by (★)
$$\begin{aligned}
|UOHam(K_{n,m})| &= \frac{|OHam(K_{n,m})|}{2|V(K_{n,m})|} \\
&= \frac{2(n!)^2}{2(2n)} \\
&= \boxed{\frac{(n!)^2}{2n}}
\end{aligned}$$

4ii. • Recall from Lecture

Fact. • G graph

• G connected & $\forall v \in V(G)$ $\deg_G(v)$ even

$\iff$ G Eulerian

• Let $n \in \mathbb{N}$

• Note K_n connected (clearly)

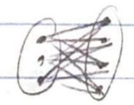
• Note K_n is $(n-1)$-regular.

That is, $\forall v \in V(K_n)$, $\deg_{K_n}(v) = n-1$

• By Fact above, $\boxed{K_n \text{ Eulerian} \iff n-1 \text{ even} \iff n \text{ odd}}$

• Let $n, m \in \mathbb{N}$

• Note $K_{n,m}$ connected (clearly)



• Recall $K_{n,m} = (A \cup B, \{\{v_i, u_j\} : i \in [n]\ j \in [m]\})$

where $A = \{v_i : i \in [n]\}$

and $B = \{u_i : i \in [m]\}$

• Note $\forall v \in A$, $\deg_{K_n}(v) = m$, and $\forall v \in B$, $\deg_{K_n}(v) = n$

• Thus $\boxed{K_{n,m} \text{ Eulerian} \iff n \text{ even and } m \text{ even}}$

(by Fact above)

4 Problem 4 **18.5 / 20**

- **0 pts** Full credit
- **1.5 pts** Forgot to divide by 2 in either part of [i]; we could traverse a Hamiltonian cycle in either one of two directions.

✓ **- 1 pts** **Forgot small cases; either n = 1, 2 for K_\[n] or n = m = 1 for K_{n, m}**

✓ **- 0.5 pts** **Forgot small cases for both parts of [i]**

- **3 pts** In the second half of part [i]; if n is not equal to m, we get no Hamiltonian cycles in this way.
- **10 pts** missing part [ii] - Eulerianness
- **2 pts** Error in part [ii]

2 this is not true for n = 1

3 this is not true for n = 1, 2

5. · Pf. · Let $G = (V, E)$ connected graph
and $H = (V, E')$ with $E' = E \cup \{\{v, w\} \in \binom{V}{2} : d_G(v, w) = 2\}$

· Let $v \in V(H) = V$
· Consider $H - v$

· Let $u, w \in V(H - v) = V \setminus \{v\}$

· Since G connected, and $u, w \in V \setminus \{v\} \subseteq V(G)$,
$\exists$ path P in G from u to w
· Let P be the shortest path in G from u to w

· Case 1: $v \in P$
· Denote $P = (v_0 = u, v_1, \dots v_{i-1}, v_i = v, v_{i+1}, \dots v_k = w)$

· Note $d_G(v_{i-1}, v_{i+1}) \leq 2$
· If $d_G(v_{i-1}, v_{i+1}) = 1$ then P is not the shortest path from u to w in G, contradiction ↯
· Thus $d_G(v_{i-1}, v_{i+1}) = 2$

· Then $\{v_{i-1}, v_{i+1}\} \in E' = E(H)$

· Thus $P' = (v_0, v_1, \dots v_{i-1}, v_{i+1}, \dots v_k)$
is a path from u to w in H. (since $E(H) \supseteq E(G)$)

· Note $v \notin P'$, so P' is a path from u to w in $H - v$.

Case 2: $v \notin P$

- Then P is a path from u to w in $H-v$ (since $E(H) \supseteq E(G)$ and $v \notin P$)

- Therefore H is 2-connected by definition

(In this problem we assumed $|V(G)| \geq 3$, so $|V(H)| \geq 3$.
The problem did not state it, but
it is a necessary condition for H to be 2-connected)

5 Problem 5 **20 / 20**

✓ **- 0 pts** **Full credit**

- 5 pts Proof has correct ideas, but a large gap.

- 6 pts Proves - correctly - that all vertices are contained in a cycle in H. But this does not imply 2-connectedness.

- 10 pts incorrect proof

6. · Recall from lecture

<u>Fact.</u> T tree $\Leftrightarrow$ T connected & T has no cycles

$\Leftrightarrow$ T connected & $|V| = |E| + 1$

6a. · $\boxed{1}$ trees T s.t. $|V| = 3$ (up to isomorphism)

6b. · $\boxed{2}$ trees T s.t. $|V| = 4$ (up to iso.)

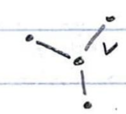
6c. • Let T tree st $|E| = 6$ and T has exactly 3 leaves

• If $\deg_T(v) \leq 2$ $\forall v \in V(T)$ then T is a path, so T has 2 leaves $\lightning$

• Thus $\deg_T(v) \geq 3$ for some $v \in V(T)$ (Fix such a v)

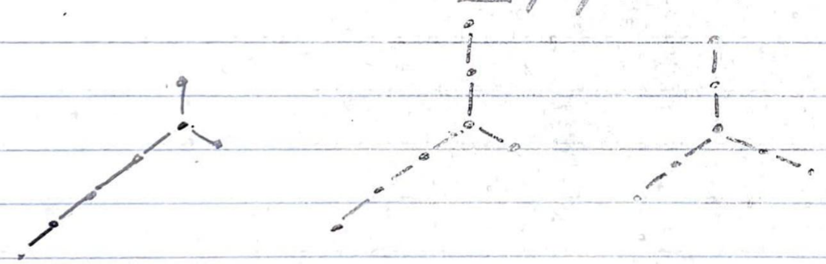
• Note $\deg_T(v) \geq 4 \Rightarrow T$ has at least 4 leaves. $\lightning$

• Thus $\deg_T(v) = 3$



• Note $\deg_T(u) \leq 2$ $\forall u \in V(T-v)$ (since otherwise T has > 3 leaves $\lightning$)

• Thus T is one of these $\boxed{3}$ graphs (up to isomorphism)



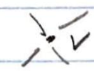
6d. • Let T tree s.t. $|E| = 6$ and T has exactly 4 leaves

• $\deg_T(v) \geq 3$ for some $v \in V(T)$ (otherwise T path, so T has 2 leaves $\lightning$)

(Fix such a v.)

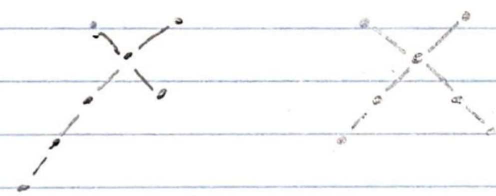
• $\deg_T(v) \leq 4$ (otherwise T has at least 5 leaves $\lightning$)

• <u>Case 1:</u> $\deg_T(v) = 4$



• Then $\deg_T(u) \leq 2 \; \forall u \in V(T-v)$ (otherwise T has > 4 leaves $\lightning$)

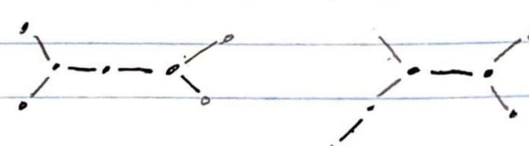
• Thus T is one of these 2 graphs (up to iso)



• <u>Case 2:</u> $\deg_T(v) = 3$

• Then $\exists ! u \in V(T-v)$ s.t. $\deg_T(v) = 3$
(if $\nexists u$, then T has 3 leaves $\lightning$
if u not unique, then T has > 4 leaves $\lightning$)

• Thus T is one of these 2 graphs (up to iso)



• Therefore there are $\boxed{4}$ such T.

6 Problem 6 **20 / 20**

- ✓ **- 0 pts** Correct
- **- 3 pts** (c) wrong answer: the correct answer is 3
- **- 2 pts** (c) justification missing
- **- 3 pts** (d) wrong answer: the correct answer is 4
- **- 2 pts** (d) justification missing
- **- 1 pts** (c) insufficient justification
- **- 1 pts** (d) insufficient justification

7. • Recall from lecture

Fact. T tree $\Leftrightarrow$ $|V(T)| = |E(T)| + 1$

• Let $m, n \in \mathbb{N}$ s.t. $K_{m,n}$ has a spanning tree whose complement inside $K_{m,n}$ is also a spanning tree of $K_{m,n}$

- Let T be such a tree, and T' be its complement in $K_{m,n}$

• Note $|V(T)| = |V(T')| = |V(K_{m,n})| = m+n$

• By fact, $|E(T)| = |E(T')| = m+n-1$

• Also, $|E(T')| = |E(K_{m,n}) \setminus E(T)|$
$= |E(K_{m,n})| - |E(T)|$

so $|E(T)| + |E(T')| = |E(K_{m,n})|$

$\Rightarrow$ $2m+2n-2 = mn$

$\Rightarrow$ $mn - 2m - 2n + 4 = 2$

$\Rightarrow$ $(m-2)(n-2) = 2$

$\Rightarrow$ $m-2=1,\ n-2=2$ or $m-2=2\ \ n-2=1$ (since $m, n \in \mathbb{N}$)

$\Rightarrow$ $\boxed{m=3,\ n=4 \text{ or } m=4,\ n=3}$

- It remains to show this choice of m, n work.

This is clear by inspection:

T T'

(note both of these are spanning trees of $K_{3,4}$)

7 Problem 7 **20 / 20**

✓ **- 0 pts** **Correct**

- 5 pts wrong answer: the correct answer is m=4, n=3 or m=3, n=4

- 5 pts incorrect proof

- 3 pts Did not actually provide an example of such a tree

- 3 pts a *spanning* treee must use all vertices

8. - Let $n \geq 3$, G_n, G'_n be the graphs in the problem statement.

8i. • $\boxed{G_n \text{ planar } \forall n \geq 3}$

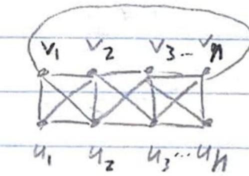
Pf: - Remove the slanted-upwards arcs from G_n's drawing and place them on the outside, looped around eachother.

• The result is a planar drawing of G_n

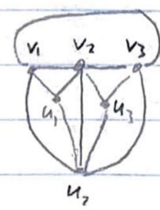
• Here is $n=6$:

8ii. • $\boxed{G'_n \text{ planar only when } n=3}$

• Pf: • Label the vertices of G'_n like so:



• Note G'_3 can be drawn:



so G'_3 planar

• However, when $n \geq 4$, a subdivision of $K_{3,3}$ exists as a subgraph of G'_n.

• Let $A = \{v_2, u_2, v_n\}$ and $B = \{v_3, u_3, v_1\}$

• Let $X = \{\{v_1, v_2\}, \{v_1, u_2\}, \{v_1, v_n\},$
$\{v_3, v_2\}, \{v_3, u_2\}, (v_3, v_4, \dots v_n),$
$\{u_3, v_2\}, \{u_3, u_2\}, (u_3, u_4, \dots u_n, v_n)\}$

- Note X is the set of edges in $K_{3,3}$ with parts A, B where some of the edges have been subdivided into paths.

- Thus G_n has been shown to have a subdivision of $K_{3,3}$ as a subgraph $\forall n \geq 4$.

- Recall from lecture that:

  Thm. G nonplanar $\Leftrightarrow$ $\exists$ subdivision H of $K_{3,3}$ or K_5 in G

- Thus by Thm, G_n nonplanar $n \geq 4$

8 Problem 8 **20 / 20**

- ✓ **- 0 pts** Correct
- **- 5 pts** (i) wrong answer [Correct answer is "for all n"]
- **- 5 pts** (ii) wrong answer [Correct answer is "only for n=3"]
- **- 5 pts** (ii) Did not prove non-planarity for n>3
- **- 3 pts** (ii) insufficient justification
- **- 2 pts** (ii) insufficient justification
- **- 5 pts** (i) incorrect proof
- **- 5 pts** (ii) incorrect proof

9. - Let G planar graph w/ $|V(G)| \geq 3$

- Let $k = \#\{v \in V(G) : \deg_G(v) \leq 5\}$

- Note $2|E(G)| = \sum_{v \in V(G)} \deg_G(v)$

$$\geq 6(|V(G)| - k)$$

$$\Rightarrow |E(G)| \geq 3|V(G)| - 3k$$

- By a theorem from lecture, since $|V(G)| \geq 3$ and G planar,

$$|E(G)| \leq 3|V(G)| - 6$$

- Thus $3|V(G)| - 3k \leq 3|V(G)| - 6$

$$\Rightarrow 3k \geq 6$$

$$\Rightarrow k \geq 2$$

- Therefore $\boxed{\text{(a) and (b) are True}}$

$\boxed{\text{True}}$

9e. • Suppose $k = 2$. Let v_1, v_2 be the two vertices w/ degree ≤ 5
• Let $d_1 = \deg_G(v_1)$ and $d_2 = \deg_G(v_2)$
• Then $|E(G)| = \frac{1}{2} \sum_{v \in V(G)} \deg_G v = \frac{1}{2}(6|V(G)| - 12 + d_1 + d_2)$
• And $|E(G)| \leq 3|V(G)| - 6$ (since G planar)

• Thus $\frac{1}{2}(d_1 + d_2) \leq 0$, so $d_1 = d_2 = 0$

• Thus $G - v_1 - v_2$ is a planar graph with <u>no</u> vertices of degree ≤ 5 (since removing vertices of degree 0 does not change the degree of any other vertices)

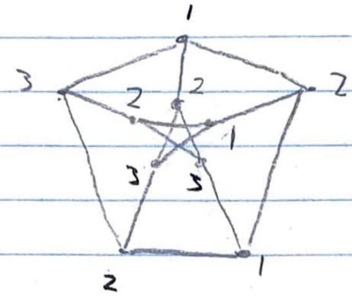
• This is a contradiction of part a.

• Thus $k \geq 3$ (since $k \neq 2$ and $k \geq 2$)

9 Problem 9 **20 / 20**

- ✓ **- 0 pts** **Correct**
- **- 5 pts** (c) wrong answer [Correct answer: True]
- **- 8 pts** (b,c) wrong answer [Correct answer: True for both]
- **- 5 pts** (c) incorrect proof
- **- 12 pts** (b,c) missing

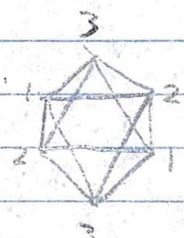
10a. • From the labeling to the right, we see $\chi(G) \leq 3$



• Also, Note $C_5 \subseteq G$ (the outer vertices)
Since C_5 is an odd cycle, $\chi(C_5) = 3$.
Thus $\chi(G) \geq \chi(C_5) = 3$.

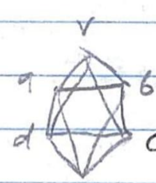
• Therefore $\boxed{\chi(G) = 3}$

10b. • From the labeling to the right, we see $\chi(G) \leq 3$



• Let $c: V(G) \to \mathbb{N}$ be a minimal coloring of G (using least # of colors)

• Label G like so:



• Note $H = \{a, b, c, d\}$ is an even cycle, so $\chi(H) = 2$.

• Note v is connected to all $v \in H$ by an edge, so $c(v) \notin \{c(u) : u \in H\}$

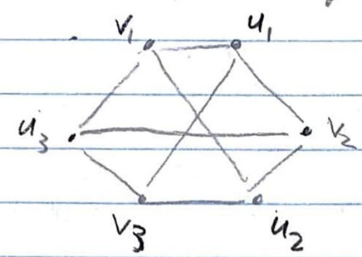
• But $\#\{c(u) : u \in H\} \geq \chi(H) = 2$.

• Thus $\chi(G) = \#\{c(u) : u \in V(G)\} \geq 3$

• Therefore $\boxed{\chi(G) = 3}$

10c. • Note $G \cong K_{3,3}$

This is clear from the following labeling:



• Claim: $\chi(K_{n,m}) = 2 \quad \forall n,m \in \mathbb{N}$

Pf: • Let $A = \{v_i : i \in [n]\}$ and $B = \{u_i : i \in [m]\}$

• Since $|E(K_{n,m})| \geq 1$, $\chi(K_{n,m}) \geq 2$

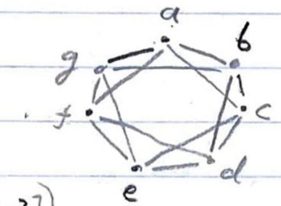
• The following is a valid 2-coloring:

$$c(v) = \begin{cases} 1 & v \in A \\ 2 & v \in B \end{cases}$$

• By claim, $\boxed{\chi(G) = 2.}$

10d. • Suppose there exists some 3-coloring c of G, $c: V(G) \to [3]$

- Label G as follows:



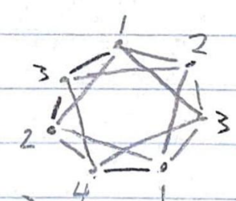
• WLOG, $c(a)=1$, $c(b)=2$
• Then $c(g)=3$ (since $K_3 \cong G[\{a,b,c\}]$) and $c(c)=3$

• Since $K_3 \cong G[\{e,f,g\}]$, $\{c(e), c(f)\} = \{1,2\}$ (since $c(g)=3$)
• Similarly, since $K_3 \cong G[\{c,d,e\}]$, $\{c(e), c(d)\} = \{1,2\}$ (since $c(c)=3$)

• Thus $\{c(e), c(d), c(f)\} = \{1,2\}$.
• But $K_3 \cong G[\{e,d,f\}]$, a contradiction ↯

• Thus $\chi(G) \geq 4$.

∴ From the coloring to the right, $\boxed{\chi(G) = 4}$

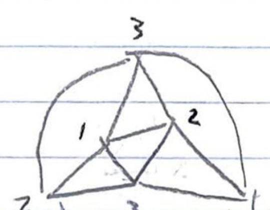


10e. • Note $C_3 \subseteq G$ (the outer vertices)

- Since C_3 is an odd cycle, $\chi(C_3) = 3$

• Thus $\chi(G) \geq \chi(C_3) = 3$

• The coloring to the right shows $\chi(G) \leq 3$

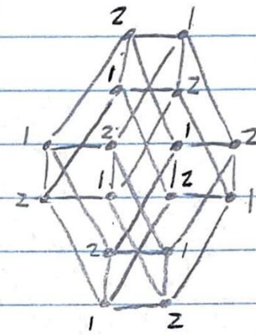


• Therefore $\boxed{\chi(G) = 3}$

10f. • Note G is a tesseract!

• Since $E(G) \neq \emptyset$, $\chi(G) \geq 2$

• From the labeling to the right, $\chi(G) \leq 2$



• Therefore $\boxed{\chi(G) = 2}$

(This can also be seen from the fact that the recursive operation that builds the tesseract preserves $\chi = 2$)

10 Problem 10 **20 / 20**

- ✓ **- 0 pts** **Correct**
- **- 7 pts** justification missing: why are these numbers minimal possible?
- **- 3 pts** colorings not shown
- **- 3 pts** (a) wrong answer [Correct answer: 3]