Math 180 Midterm 1 (Fall 2015)

Some estimates and formulas:

$$n^{n/2} \leq n! \leq \left(\frac{n+1}{2}\right)^n, \qquad n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n, \qquad e = 2.71\ldots \cong 19/7$$

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k, \qquad \frac{2^n}{n+1} \leq \binom{n}{\lfloor n/2 \rfloor} \leq 2^n$$

Problem 1. Find the number of:

(a) permutations $(\sigma_1, \sigma_2, \ldots, \sigma_8)$ with $\sigma_1 + \sigma_2 = 8$,

(b) the number of subsets of $\{1, 2, \ldots, 8\}$ whose minimum or maximum is 4,

(c) the number of integer solutions $y_1 + x_2 + x_3 = 8$ where $y_1 > 0$ and $x_2, x_3 \geq 0$.

Problem 2. Recall that the number of graphs with vertices $\{1, 2, \ldots, n\}$ is $g(n) = 2^{\binom{n}{2}}$.

(a) What is the number of bipartite graphs with bipartition $V_1 = \{1, 2, \ldots, k\}$ and $V_2 = \{k+1, k+2, \ldots, n\}$.

(b) Show that the number $b(n)$ of bipartite graphs with vertices $\{1, 2, \ldots, n\}$ is $b(n) = \sum_{k=0}^{n} \binom{n}{k} 2^{k(n-k)}$.

(c) Use estimates to give an upper bound to the number $b(n)$ of bipartite graphs with vertices $\{1, 2, \ldots, n\}$ of the form $2^{f(n)}$ for some function $f(n)$.

(d) Use the estimate in (c) to show that $b(n) = o(g(n))$ (i.e. $\lim_{n\to\infty} b(n)/g(n) = 0$).

(e) Very briefly, what does $b(n) = o(g(n))$ say about graphs?

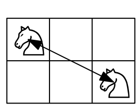
Problem 3. For (b)(c)(d), draw all simple graphs with given score (up to isomorphism) and prove that no other such graphs exist.

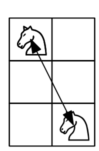
(a) Is $D = (1, 2, 3, 4, 5, 6, 6, 7)$ a degree sequence?

(b) $D = (1, 1, 2, 2, 2, 2)$,

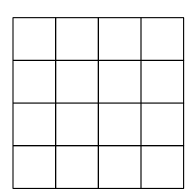
(c) $D = (1, 1, 1, 5, 5, 5)$.

Problem 4. Recall that a *knight* ♘ from chess has the following moves:

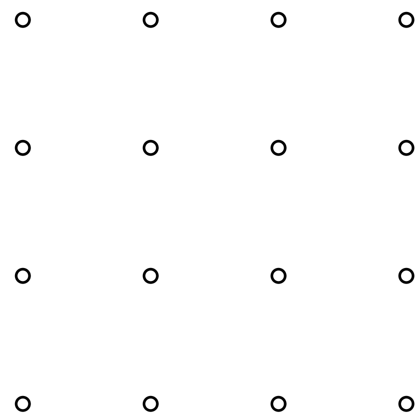




4×4 board:



(a) Draw a graph that encodes the cells from a 4×4 board and the possible knight moves.



(b) Decide if it is possible for the knight to move around the 4×4 board so that it makes every possible move exactly once.

(c) *Bonus (1 point):* Decide if it is possible for the knight to move around the 4×4 board so that every cell is visited exactly once.

Problem 5. True or False Circle the answers only **with ink**, next to the questions. No reasoning/calculations will be taken into account.

(a) If a person is paid every other Friday, there is a month in the year with three payments. **T. or F.**

(b) The name EMMETT has more than 88 rearrangements of its letters. **T. or F.**

(c) A prime number p divides all the binomial numbers $\binom{p}{1}, \binom{p}{2}, \ldots, \binom{p}{p-1}$. **T. or F.**

(d) There are more injections than surjections from $\{A, B, C, D\}$ to $\{1, 2, 3, 4\}$. **T. or F.**

(e) $n! \sim (2n)(2n-2)(2n-4) \cdots 4 \cdot 2$. **T. or F.**

(f) There are more subsets of $\{1, 2, \ldots, 11\}$ of odd size than even size. **T. or F.**

(g) There are the same number of nonnegative integer solutions to $x_1 + x_2 + x_3 = 4$ as positive ingeter solutions to $y_1 + y_2 + y_3 = 7$. **T. or F.**

(h) All complete graphs K_n $(n > 1)$ are Eulerian. **T. or F.**

(i) All complete graphs K_n $(n > 1)$ have Hamiltonian cycles. **T. or F.**

(j) Since graph isomorphisms is a hard problem then $2^{\binom{n}{2}}/n!$ does not approximate the number of graphs. **T. or F.**