Math 180: Graph theory. Midterm

Instructor: Damir Yeliussizov

Exam time: 14:00-14:50 PM, Apr 27, 2018

Last name: Berdy First name: Aaron

Student ID: [redacted]

There are 5 problems.
No books, notes, calculators, phones, conversations, etc.
Turn off your cell phones.

P 1 (10)	P 2 (10)	P 3 (10)	P 4 (10)	P 5 (10)	Total (50 pt)
10	10	10	10	10	50

Problem 1. (10 points) Find the number of pairs (A, B) of subsets $A, B \subseteq \{1, \ldots, n\}$ such that $A \cap B = \emptyset$ (i.e. disjoint). (Explain your answer. Note: count (X, Y) and (Y, X) as different pairs.)

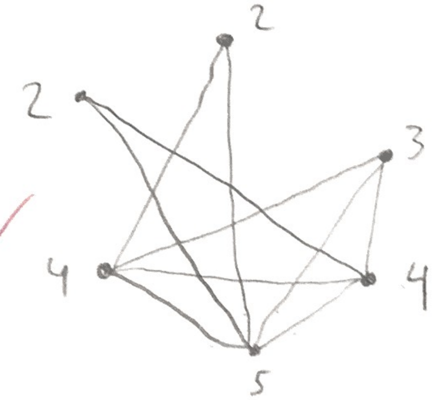
for every A, $|A| = k$, there are 2^{n-k} possible options for B, since $B \subseteq \{1, \ldots, n\} \setminus A$, which has size $n-k$.

$$\sum_{k=0}^{n} \binom{n}{k} 2^{n-k} = \sum_{k=0}^{n} \binom{n}{k} 1^k 2^{n-k} = (1+2)^n = 3^n$$

Problem 2. (10 points) Decide whether these are graph degree sequences:
(a) $(1, 2, 3, 4, 4, 5)$
(b) $(2, 2, 3, 4, 4, 5)$
(c) $(2, 2, 2, 4, 5, 5)$
(Explain your answers. If yes, then show a graph.)

a) no. $\sum_{v \in V} \deg(v) = 2|E| = 19$, but $|E|$ cannot be $\frac{19}{2}$.

b)



here is an example, w $\deg(v)$ labeled for all vertices.

c) no. $\exists G_1 : (2,2,2,4,5,5) \iff \exists G_2 : (1,1,1,3,4) \iff \exists G_3 : (0,0,0,2)$, and clearly there cannot exist a simple graph with one vertex of degree > 0 and the rest $\deg = 0$.

Problem 3. (10 points) Let $G = (V, E)$ be an undirected simple graph on n vertices satisfying the condition $\deg(u) + \deg(v) \geq n$ for all vertices $u \neq v \in V$. Show that G is connected.

Proof by contradiction:

assume G is not connected.

pick any 2 vertices $\in V$, u and v. let S be the set of all vertices adjacent to u or v. Since the neighbors of u and v are disjoint (by assumption), $|S| = \deg(u) + \deg(v) \geq n$. By assumption, $u \notin$ neighbors(v), $v \notin$ neighbors(u), and no loops, so $u, v \notin S$. Therefore, the maximum size of S is $n-2$.

This is a contradiction, since $|S| = \deg(u) + \deg(v) \geq n > n-2$,

Therefore the assumption must be false and G is connected. □

Problem 4. (10 points) Show that in every connected undirected graph on n vertices we can add at most $n/2$ new edges so that the resulting graph will have an Eulerian cycle. (Multiple edges are allowed.)

A graph, G, has an Eulerian cycle iff deg(v) is even, for all $v \in V$.

There are an even number of vertices with odd degrees, since $\sum_{v \in V} \deg(v) = 2|E|$, which is even.

To modify the graph to have an Eulerian cycle, partition the set of vertices with odd degrees into pairs and then add an edge from u to v for each pair (u, v).

Since there are at most n vertices with odd degree, there are at most $\frac{n}{2}$ pairs of odd vertices, and therefore at most $\frac{n}{2}$ new edges you must add.

The degree of all previously odd-degree vertices will increase by 1, so all vertices will have an even degree $\Rightarrow$

G will now have an Eulerian cycle. □

Problem 5. (10 points) Prove that a graph is bipartite if and only if it has no cycles of odd length.

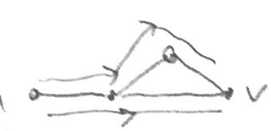
$\Rightarrow$

let G be a bipartite graph, A, B two sets s.t. $A \cup B = V$, $A \cap B = \emptyset$. Let $f(v) = \begin{cases} 1, & \text{if } v \in A, \\ 0, & \text{otherwise} = v \in B \end{cases}$. Note that $f(u) \neq f(v)$

$\forall u,v \in A, (u,v) \notin E$; $\forall u,v \in B, (u,v) \notin E$.

for any u, v s.t. $(u,v) \in E$, since that would imply an edge between 2 vertices in A or 2 vertices in B. Assume for contradiction that an odd-length cycle exists. Then moving along the cycle, $f(v)$ would change an odd number of times between 0 and 1, and then $f(v) = 1 = 0$ could be shown for any v in the cycle, which is a contradiction. $\therefore$ no odd-length cycles exist $\square$

$\Leftarrow$ let G be a graph with no cycles of odd length. Then, between any 2 vertices, u and v, all paths $u \to v$ are even length, or all are odd. Otherwise one could follow an even path one way, and the odd one back and find an odd cycle. if the paths share edges, only consider the parts before and after the "fork":



Choose a vertex $v_A \in V$. let A be the set of vertices with an even distance from v_A (and connected to v_A). let B be the set of vertices w/ an odd distance from v_A or not connected to v_A. A and B are disjoint, and $\forall v_1, v_2 \in A$ or $v_1, v_2 \in B$, all paths from v_1 to v_2 will be even, since otherwise $v_A \xrightarrow{even} v_1 \xrightarrow{odd} v_2$ or $v_A \xrightarrow{odd} v_1 \xrightarrow{odd} v_2$ would be a contradiction with $v_A \xrightarrow{even} v_2$ or $v_A \xrightarrow{odd} v_2$, respectively. rest on back

Since $\nexists$ any odd paths between $v_1, v_2 \in A$ or $v_1, v_2 \in B$, v_1 cannot be adjacent to v_2,

$\Rightarrow (v_1, v_2) \notin E \quad \forall v_1, v_2 \in A$ or $\forall v_1, v_2 \in B$

$\therefore$ G is bipartite $\square$

$\therefore$ G is bipartite iff $\nexists$ any cycles of odd length $\square$

excellent.