Math 174E
Spring 2016
Midterm
05/06/16
Time Limit: 50 Minutes

Name(Print): Solution.

UID: __________

Signature: __________

This exam contains 9 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may only use your cheating sheet and an non-graphic calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	10	
3	20	
4	20	
5	20	
6	10	
Total:	100	

1. (20 points) Define these terms in about two or three sentences each, be as complete as possible.

   (a) short selling

   the sale of a security that is not owned by the seller, or that the seller has borrowed.

   (b) put option

   An option contract give the ~~ower~~ owner the right, not obligation, to sell a specified amount of an underlying security at a specified price within a specified time.

   (c) forward contract

   A forward contract is a customized contract between two parties to buy or sell an asset at a specified price on a future date.

   (d) arbitrage

   The simultaneous ~~ph~~ purchase and sale of an asset in order to profit from a difference in the price.

2. (a) (5 points) If R_1 and R_2 are the zero rates(with continuous compounding) for maturities T_1 and T_2, what is the forward interest rate for the period of time between T_1 and T_2 ? Please show your analytical derivation of this formula.

$$Ae^{R_1T_1} \cdot e^{R_F(T_2-T_1)} = Ae^{R_2T_2}$$

$$R_1T_1 + R_F(T_2-T_1) = R_2T_2$$

$$R_F = \frac{R_2T_2 - R_1T_1}{T_2-T_1}.$$

(b) (5 points) The one year zero rate is 6% and two year zero rate is 8%. What is the one year forward rate?

$R_1 = 6\%$ $\quad R_2 = 8\%$

$T_1 = 1$ $\quad T_2 = 2$

$$R_F = \frac{8\% \cdot 2 - 6\% \cdot 1}{2-1} = 10\%$$

3. Let c and p be the price of European call and put options respectively at time $t = 0$. Let K be the exercise (or strike) price at the time of maturity $t = T$ and S_0 be the price of the asset at time $t = 0$.

(a) (6 points) State the put-call parity relation and use it to deduce the inequalities:

$$c \geq S_0 - Ke^{-rT} \quad \text{and} \quad p \geq Ke^{-rT} - S_0.$$

put-call parity : $c + Ke^{-rT} = p + S_0$

$c = p + S_0 - Ke^{-rT}$

$\geq S_0 - Ke^{-rT}$

as $p \geq 0$.

$p = c + Ke^{-rT} - S_0$

$\geq Ke^{-rT} - S_0$

as $c \geq 0$.

(b) (7 points) Show, using an no-arbitrage argument, that $c \leq S_0$.

Consider two portfolios:

A: one call option

B: one share of stock

At $t = T$, the value of two portfolios,

$S_T \geq K$, $V_A = S_T - K \leq V_B = S_T$

$S_T < K$ $V_A = 0 \leq V_B = S_T$

i.e. $V_A \leq V_B$ in any case. Thus $c \leq S_0$ at $t = 0$ by no-arbitrage argument.

(c) (7 points) Use the result form part (b) along with the put-call parity to deduce the inequalities

$$Ke^{-rT} - S_0 \le p \le Ke^{-rT}.$$

$p \geq Ke^{-rT} - S_0$ from part (a)

By put-call parity

$$c + Ke^{-rT} = p + S_0$$

$$p = c - S_0 + Ke^{-rT}$$

$$= (c - S_0) + Ke^{-rT}$$

$$\le Ke^{-rT}$$ as $c \le S_0$ from part (b).

$$\Rightarrow Ke^{-rT} - S_0 \le p \le Ke^{-rT}.$$

4. (a) (15 points) By setting up an appropriate riskless portfolio, derive the formula for pricing an option on a 1-step binomial tree, carefully explaining any ideas that you might use. The initial price is S_0 and the price changes to either uS_0 or dS_0, with corresponding option payoffs f_u and f_d respectively.

$$\begin{tikzcd}[column sep=large] & \Delta \cdot u S_0 - f_u \\ \Delta \cdot S_0 - f \arrow[ur, no head] \arrow[dr, no head] & \\ & \Delta \cdot d S_0 - f_d \end{tikzcd}$$

portfolio: $\begin{cases} \Delta \text{ shares of stock} \\ -1 \text{ option} \end{cases}$

To make the portfolio riskless, we have

$$\Delta \cdot u S_0 - f_u = \Delta \cdot d S_0 - f_d$$

$$\Delta \cdot (u-d) S_0 = f_u - f_d.$$

$$\Delta = S_0 \frac{f_u - f_d}{u-d}.$$

riskless portfolio earn risk-free rate.

$$e^{rT}(\Delta S_0 - f) = \Delta \cdot u S_0 - f_u.$$

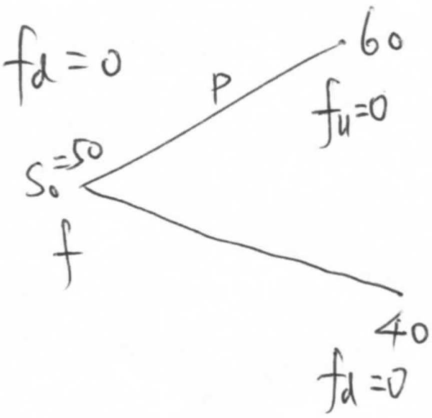
$$f = \Delta S_0 - (\Delta u S_0 - f_u) e^{-rT}$$

$$= [p f_u + (1-p) f_d] e^{-rT}$$

where $p = \dfrac{e^{rT} - d}{u-d}.$

(b) (5 points) Consider a European call option for a stock with initial price $S_0 = 50$, strike price $K = 70$, expiration $T = 2$(years) and risk-free interest rate $r = 5\%$ per year. Calculate the option price c at $t = 0$ using a 1-step binary tree model of the underlying stock S with up and down factors $u = 1.2$ and $d = 0.8$ and real probability $p^* = 0.6$ for an up step.

$S_0 = 50$, $K = 70$, $f_u = 0$, $f_d = 0$



$$f = e^{-rT}[p f_u + (1-p) f_d]$$

$= 0$

5. (a) (10 points) Suppose that p_1, p_2, p_3 are the prices of European put options with strike prices K_1, K_2, K_3 respectively, where $K_3 > K_2 > K_1$ and $K_3 - K_2 = K_2 - K_1$. All options have the same maturity. Prove that the following inequality holds

$$p_2 \leq \frac{p_1 + p_3}{2}.$$

K_1 K_2 K_3

Consider a portfolio with a long put option with strike price K_1, a long put option with strike price K_3 and 2 short put options with strike price K_2. Denote P the value of the portfolio at $t=T$

if $S_T < K_1$, $P = (K_1 - S_T) - 2(K_2 - S_T) + K_3 - S_T = 0$

if $K_1 < S_T < K_2$ $P = K_3 - S_T - 2(K_2 - S_T) = (K_3 - K_2) - (K_2 - S_T) \geq 0$

if $K_2 < S_T < K_3$ $P = K_3 - S_T \geq 0$,

if $S_T > K_3$ $P = 0$

Since $P \geq 0$ at $t=T$ is always correct, the same must holds at $t=0$, i.e. $P_1 + P_3 - 2P_2 \geq 0$

$P_2 \leq \frac{P_1 + P_3}{2}$.

(b) (10 points) Consider a "digital" option d, based on an underlying stock S and a "strike price" K, whose payout at the expiration T is

$$d(T) = \begin{cases} 1 & \text{if } S(T) \geq K \\ 0 & \text{if } S(T) < K. \end{cases}$$

Use a no-arbitrage argument to show that the value $d(t)$ satisfies $d(t) \leq e^{-(T-t)}$.

consider two portfolios:

A: one digit option d

B: a bond with payout $=1$.

At $t=T$, the value of these two ~~potio~~ portfolios are

$S_T \geq K$, $V_A = 1 \leq V_B = 1$.

$S_T < K$, $V_A = 0 < V_B = 1$

since $V_A \leq V_B$ at $t=T$ is always correct, the same must holds at t.

i.e. $d(t) \leq e^{-r(T-t)}$

6. (10 points) Let x_n be a random walk with binomial increments, i.e.,

$$\begin{aligned} x_{n+1} &= x_n \pm 1 \\ x_0 &= 0 \end{aligned}$$

with equal probability of going up (+) or down (−). Find the probability $P(x_{20} \leq 17)$.

$$P(X_{20} \leq 17) = 1 - P(X_{20} > 17)$$

$$= 1 - P(X_{20} = 18) - P(X_{20} = 20)$$

$$= 1 - \binom{20}{19}\left(\frac{1}{2}\right)^{20} + \binom{20}{20}\left(\frac{1}{2}\right)^{20}.$$

$$= 1 - \binom{20}{19}\left(\frac{1}{2}\right)^{20} - \left(\frac{1}{2}\right)^{20}$$

$$= 1 - 21 \cdot \left(\frac{1}{2}\right)^{20}$$