Math 171: Final
2021 Winter

Instructions:

- The exam will begin on March 16th at 9AM PT. You will be given **24 hours** to complete and submit your works. The submission window will be closed on March 17th at 9AM.
- **No late submission** will be considered. Make sure to allow enough time to complete and submit your works. Make-ups for the exam are permitted only under exceptional circumstances, as outlined in the UCLA student handbook.
- The exam will be **open book/open notes**. You may also use a physical and/or online calculator. However, you must **show your works to receive credit.**
- You must **sign the code of conduct.** Any deviation from the rules may render your exam void. Also, if needed, you may be contacted after the exam and asked for additional explanations of solutions for problems on the exam.
- A Gradescope link for submitting your work will be provided on the CCLE course webpage.

Please read and sign the following honor code:

> *"I assert, on my honor, that I have not received assistance of any kind from any other person while working on the exam and that I have not used any non-permitted materials or technologies during the period of this evaluation."*

Name: ____________________

UID: ____________________

Signature: ____________________

1. (10 pts) Do the following:

(a) Belathor has two candles, labeled A and B, respectively. The lifetime of Candle A is distributed as $\mathrm{Exp}(\lambda_A)$ and the lifetime of Candle B is distributed as $\mathrm{Exp}(\lambda_B)$. Belathor lits up Candle A at time 0, and then lits up Candle B at a deterministic time $t_0 > 0$.

What is the probability that Candle A burns out before Candle B?

(b) Fix $\lambda > 0$ and $p \in (0, 1)$. Let X has a Poisson distribution with mean λ. Also, suppose that the conditional distribution of Y, given that $X = x$, is $\mathrm{Binomial}(x, p)$. What is the distribution of Y?

2. (10 pts) Let $\{N(t)\}_{t\geq 0}$ be a Poisson process with rate $\lambda > 0$. Also, let T_k denote the time of the kth arrival.

(a) Find $\mathbf{P}(N(3) = 1, N(5) = 4)$.

(b) Compute $\mathbf{P}(T_1 > 4 \mid T_3 > 5)$.

(c) Compute $\mathbf{Cov}(N(6), N(4) - N(1))$.

3. (10 pts) Sakaki wants to mine 3 cryptocurrency blocks using two computers. Each computer can mine a single block at a time. The time Computer 1 (respectively, Computer 2) takes to mine a block is exponentially distributed with mean of 5 minutes (respectively, 10 minutes), independently of everything else.

Compute the expected time until all the three blocks are mined, under each of the following scenarios:

(a) Assign two blocks to Computer 1 and one block to Computer 2.

(b) Assign one block to each of Computer 1 and 2, and then assign the remaining third block to whichever computer that finishes mining first.

4. (10 pts) On a given day, starting from 5 PM, spectators arrive at a concert hall according to a Poisson process with rate $\lambda = 16$ per minutes. Suppose that the arriving spectators have to wait until 8 PM of that day to enter the hall. Let X be the total number of spectators who have arrived by 8 PM, and let W be the sum of the waiting times for all spectators.

(a) For each $k \geq 0$, compute $\mathbf{E}[W \mid X = k]$.

(*Hint:* Use the idea of conditioning.)

(b) What is the value of $\mathbf{E}[W]$?

(c) Now, suppose that 15% of the spectators are VIP and thus allowed to enter the hall at 7 PM. In particular, any VIP spectators arriving after 7 PM do not have to wait at all. (Of course, the remaining 85% of the spectators still have to wait until 8 PM.) What is the value of $\mathbf{E}[W]$ in this case?

5. (10 pts) At time $t = 0$, Madesi installs a light lamp that uses one light bulb at a time. Madesi will replace any failed bulb immediately. Each time Madesi installs a new bulb, it will be chosen from type-1 bulbs with probability $3/5$ and type-2 bulbs with probability $2/5$, independently of everything else. The lifetime of each bulb is exponentially distributed with mean of 2 years, regardless of the type.

(a) Given that there are no failures up to time t (in years), what is the expected time between 0 and the failure of the first type-1 bulb?

(b) Just after the 10th failure, what is the probability that exactly four of type-2 bulbs have failed up to that time?

(c) Find the probability that four failures from type-2 bulbs occur before two failures from type-1 bulbs occur.

(d) Compute the limit, as $t \to \infty$, of the probability that type-2 bulbs are used at time t.

6. (10 pts) Tourists arrive at a tourist spot according to a Poisson process with rate λ per hour. When a new tourist arrive at the spot, any previous tourist is immediately escorted outside. Suppose that the time needed to complete the tour is exponentially distributed with mean 10 minutes, independently of everything else.

(a) What fraction of tourists will complete the tour (before they are escorted outside)?

(b) Determine the proportion of the time in the long run that there is no one in the tourist spot.

(c) Hadvar visits the tourist spot. As he is a renowned veteran, the tourist office decided to provide an accommodation: he will not be escorted outside unless three or more new tourists arrive during his tour. Find the probability that Hadvar completes the tour.

7. (10 pts) Students arrive at Professor Baek's office according to a Poisson process with rate 5 per hour. Each arriving student will leave immediately if Baek is busy; otherwise he/she will stay in Baek's office and ask questions for an amount of time (in minutes) that is uniformly distributed between 6 and 30, independently of all the others.

(a) Find the long-run proportion of time at which students are in Baek's office.

(b) Determine the limiting proportion of arriving students who stay in Baek's office.

8. (10 pts) Let $\{N(t)\}_{t\geq 0}$ be a Poisson process with rate 1, and let $\lambda(\cdot)$ be a non-negative function such that $\mu(t) = \int_0^t \lambda(s)\,\mathrm{d}s$ is finite for any $t \geq 0$. Show that $\{\tilde{N}(t)\}_{t\geq 0}$ defined by

$$\tilde{N}(t) = N(\mu(t))$$

is a Poisson process with rate function $\lambda(\cdot)$.

9. (10 pts) Let $\{N(t)\}_{t\geq 0}$ be the counting process associated to an arrival process, and let $\lambda > 0$ be constant. Suppose that the followings hold:

i) $N(0) = 0$.

ii) *(Independent Increments)* For any $0 \leq t_0 \leq t_1 \leq \cdots \leq t_n$, the increments $\{N(t_k) - N(t_{k-1})\}_{1\leq k\leq n}$ are independent.

iii) For any $t \geq 0$, we have $\lim_{h\to 0^+} \mathbf{P}(N(t+h) - N(t) = 1)/h = \lambda$.

iv) For any $t \geq 0$, we have $\lim_{h\to 0^+} \mathbf{P}(N(t+h) - N(t) \geq 2)/h = 0$.

The aim of this problem is to illustrate a method of showing that $\{N(t)\}_{t\geq 0}$ is a Poisson process with rate λ. To do so, we define $g_k(t)$ for each $k = 0, 1, 2, \ldots$ by

$$g_k(t) = \mathbf{P}(N(t) = k),$$

(a) Show that $g_k(t)$ satisfies

$$g_k'(t) = \begin{cases} -\lambda g_0(t), & \text{if } k = 0, \\ -\lambda g_k(t) + \lambda g_{k-1}(t), & \text{if } k \geq 1. \end{cases}$$

(*Hint:* $g_k'(t) = \lim_{h\to 0^+}(g_k(t+h) - g_k(t))/h$.)

(b) Using the previous step, show that

$$\frac{\mathrm{d}}{\mathrm{d}t}(e^{\lambda t} g_k(t)) = \begin{cases} 0, & \text{if } k = 0, \\ \lambda e^{\lambda t} g_{k-1}(t), & \text{if } k \geq 1, \end{cases}$$

and then deduce that $g_0(t) = e^{-\lambda t}$ and $g_1(t) = (\lambda t)e^{-\lambda t}$.

21W-MATH171-1 Final

DAVID DAVINI

TOTAL POINTS

81.8 / 90

QUESTION 1

1 Question 1 **10 / 10**

Part a) : 5 points

✓ **- 0 pts** **Everything correct**

- 1 pts Minor issues

- 3 pts Incorrect application of the results regarding the exponential race.

- 5 pts Essentially no work is provided

- 0 pts There are some issues

Part b) : 5 points

✓ **- 0 pts** **Everything correct**

- 5 pts Essentially no work is provided

- 0 pts There are some issues

QUESTION 2

2 Question 2 **10 / 10**

Part a) : 4 points

✓ **- 0 pts** **Everything correct**

- 1 pts Minor mistakes

- 0 pts There are some issues

Part b) : 3 points

✓ **- 0 pts** **Everything correct**

- 1.5 pts Incorrect way of splitting the conditional probability.

- 2 pts Incorrect reasoning

- 0 pts There are some issues

Part c) : 3 points

✓ **- 0 pts** **Everything correct**

- 1 pts Minor mistakes

- 2 pts Incorrect reasoning regarding the independent increments condition.

- 0 pts There are some issues

QUESTION 3

3 Question 3 **5 / 10**

Basic knowledge

✓ **- 0 pts** **The solution is showing some understanding on the exponential race.**

- 3 pts No sign of understanding on the exponential race

Understanding of the problem

- 0 pts Perfect understanding and computation

- 1 pts Almost perfect understanding with only minor mistakes

- 3 pts Decent understanding with moderate mistakes

✓ **- 5 pts** **Largely inaccurate understanding with major mistakes**

- 7 pts Essentially no understanding is provided.

QUESTION 4

4 Question 4 **8.8 / 10**

Part a) : 5 points

✓ **- 0 pts** **Everything correct**

- 2 pts Moderate mistakes

- 3.5 pts Critical misunderstanding on the conditional distribution of waiting times given $$X = k$$

- 5 pts Essentially no work is provided

Part b) : 2.5 points

- 0 pts Everything correct (ignoring errors carried forward)

✓ **- 1.2 pts** **There are some issues**

- 2.5 pts Essentially no work is provided

Part c) : 2.5 points

✓ **- 0 pts** **Everything correct (ignoring errors carried forward)**

- 1.2 pts There are some issues

- **2.5 pts** Essentially work is provided

1 Unit mismatch. ($$t_0=3$$ hours and $$\lambda=16$$ people per minute)

2 Unit mismatch, but I did not mark off points for errors carried forward.

QUESTION 5

5 Question 5 **10 / 10**

Part a) : 2 points

✓ **- 0 pts Everything correct**

- **1 pts** Minor mistakes (such as missing the term $$t$$, computation mistakes)

- **0 pts** There are some issues

Part b) : 3 points

✓ **- 0 pts Everything correct**

- **0 pts** There are some issues

Part c) : 3 points

✓ **- 0 pts Everything correct**

- **1 pts** Minor mistakes

- **0 pts** There are some issues

Part d) : 2 points

✓ **- 0 pts Everything correct**

- **1 pts** Mistakes in applying SLLN

- **0 pts** There are some issues

QUESTION 6

6 Question 6 **8 / 10**

Part a) : 4 points

- **0 pts** Everything correct

✓ **- 1 pts Minor issues**

- **2 pts** Major issues

- **4 pts** Essentially no work is provided

Part b) : 3 points

✓ **- 0 pts Everything correct**

- **1 pts** Minor issues

- **2 pts** Major issues

- **3 pts** Essentially no work is provided

Part c) : 3 points

- **0 pts** Everything correct

✓ **- 1 pts Minor issues**

- **2 pts** Major issues

- **3 pts** Essentially no work is provided

3 10 minutes = 1/6 hours

4 Error carried forward

5 You should have $$\lambda$$

QUESTION 7

7 Question 7 **10 / 10**

Part a) : 5 points

✓ **- 0 pts Everything correct**

- **1 pts** Minor issues

- **3 pts** Major issues

- **5 pts** Essentially no work is provided

Part b) : 5 points

✓ **- 0 pts Everything correct (ignoring errors carried forward)**

- **1 pts** Minor issues

- **3 pts** Major issues

- **5 pts** There are some issues

6 $$\lambda$$ = 5 per hour = $$\frac{1}{2}$$ per minute

QUESTION 8

8 Question 8 **10 / 10**

✓ **- 0 pts Everything correct**

Solution based on the definition

- **1 pts** $$\tilde{N}(0) = 0$$ is not checked

- **3.5 pts** Poisson increment condition is not rigorously checked (i.e., the solution is not properly and explicitly demonstrating understanding on this condition.)

- **2.5 pts** Independent increments condition is not rigorously checked (i.e., the solution is not properly and explicitly demonstrating understanding on this condition.)

- **0 pts** There are some issues

- 10 pts Essentially no work is provided

QUESTION 9

9 Question 9 **10 / 10**

Verifying $$g_0'(t)=-\lambda g_0(t)$$: 6 points

✓ **- 0 pts** **Everything correct**

- 1 pts Minor issues

- 2.5 pts Moderate issues

- 4 pts Major issues

- 6 pts Essentially no work is provided

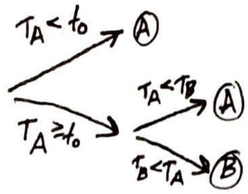
Verifying $$g_0(t)=e^{-\lambda t}$$: 4 points

✓ **- 0 pts** **Everything correct**

- 1 pts Minor issues

- 1.5 pts Verified $$(e^{\lambda t}g_0(t))' = 0$$ but did not explain why this leads to the desired solution

- 4 pts Essentially no work is provided



1a. • Let T_A, T_B be the lifetimes of the candles
• We know $T_A \sim Exp(\lambda_A)$ and $T_B \sim Exp(\lambda_B)$
• Also, Let S_A, S_B be the time from start until the candles burn out
• So $S_A = T_A$, and $S_B = T_B + t_0$

• Note $\mathbb{P}(\{\text{Candle A burns out before Candle B}\})$

$$= \mathbb{P}(S_A < S_B)$$
$$= \mathbb{P}(T_A < T_B + t_0)$$
$$= \mathbb{P}(T_A < T_B + t_0 \mid T_A < t_0)\,\mathbb{P}(T_A < t_0) + \mathbb{P}(T_A < T_B + t_0 \mid T_A \geq t_0)\,\mathbb{P}(T_A \geq t_0) \qquad (\star)$$

• Note $\mathbb{P}(T_A < T_B + t_0 \mid T_A < t_0) = 1$

and $\mathbb{P}(T_A < T_B + t_0 \mid T_A \geq t_0) = \mathbb{P}(T_A < T_B) = \frac{\lambda_A}{\lambda_A + \lambda_B}$

(lack of memory property) (min of exp's thm, from lecture)

and $\mathbb{P}(T_A < t_0) = 1 - e^{-\lambda_A t_0}$

• So $(\star) = 1 \cdot (1 - e^{-\lambda_A t_0}) + \left(\frac{\lambda_A}{\lambda_A + \lambda_B}\right) e^{-\lambda_A t_0}$

$$= \boxed{1 - \left(\frac{\lambda_B}{\lambda_A + \lambda_B}\right) e^{-\lambda_A t_0}}$$

16. $\cdot$ We have $X \sim Pois(\lambda)$ and $Y \mid X=x \sim Binom(x,p)$

$\cdot$ So $\mathbb{P}(X=x) = \frac{\lambda^x}{x!}e^{-\lambda}$ $\quad x \in \{0,1,\dots\}$
and $\mathbb{P}(Y=y \mid X=x) = \binom{x}{y} p^y q^{x-y}$ $\quad y \in \{0,1,\dots x\}$

$\cdot$ So $\mathbb{P}(Y=y) = \sum_{x=0}^{\infty} \mathbb{P}(Y=y \mid X=x)\mathbb{P}(X=x)$ (total probability)

$$= \sum_{x=y}^{\infty} \binom{x}{y} p^y q^{x-y} \frac{\lambda^x}{x!} e^{-\lambda}$$

$$= \sum_{x=y}^{\infty} \frac{\cancel{x!}}{y!(x-y)!} p^y q^{x-y} \frac{\lambda^x}{\cancel{x!}} e^{-\lambda}$$

$$= \frac{p^y e^{-\lambda} \lambda^y}{y!} \sum_{x=y}^{\infty} \frac{q^{x-y}\lambda^{x-y}}{(x-y)!}$$

$$= \frac{p^y e^{-\lambda}\lambda^y}{y!} \left(\sum_{r=0}^{\infty} \frac{q^r \lambda^r}{r!} \right)$$

$$= \frac{(\lambda p)^y e^{-\lambda}}{y!} \left(e^{\lambda q} \right)$$

$$= \frac{e^{\lambda(q-1)} (\lambda p)^y}{y!}$$

$$= \frac{(\lambda p)^y}{y!} e^{-p\lambda}$$

$\cdot$ Thus $Y \sim Pois(\lambda p)$

1 Question 1 **10 / 10**

Part a) : 5 points

✓ **- 0 pts** **Everything correct**

- 1 pts Minor issues

- 3 pts Incorrect application of the results regarding the exponential race.

- 5 pts Essentially no work is provided

- 0 pts There are some issues

Part b) : 5 points

✓ **- 0 pts** **Everything correct**

- 5 pts Essentially no work is provided

- 0 pts There are some issues

2a. • We have $N \sim PP(\lambda)$, with arrival times $\{T_k\}_{k \in \mathbb{N}}$

• For convenience, we'll denote $N_t = N(t) \quad \forall t \geq 0$

• $\mathbb{P}(N_3 = 1, N_5 = 4) = \mathbb{P}(N_5 = 4 \mid N_3 = 1)\, \mathbb{P}(N_3 = 1)$

$= \mathbb{P}(N_5 - N_3 = 3 \mid N_3 = 1)\, \mathbb{P}(N_3 = 1)$

$= \mathbb{P}(N_5 - N_3 = 3)\, \mathbb{P}(N_3 - N_0 = 1)$

(by independent increments)

$= \left(\frac{(2\lambda)^3}{3!} e^{-2\lambda}\right)\left(\frac{(3\lambda)^1}{1!} e^{-3\lambda}\right)$

$= \frac{24\lambda^4}{6} e^{-5\lambda}$

$= \boxed{4\lambda^4 e^{-5\lambda}}$

• Since $N_s - N_t \sim Pois((s-t)\lambda)$

26. • Note that $\{T_k > t\} = \{N(t) < k\} \quad \forall k \in \mathbb{N}\ \forall t \in \mathbb{R}^+$

• So $\mathbb{P}(T_1 > 4 \mid T_3 > 5) = \mathbb{P}(N(4) < 1 \mid N(5) < 3)$

$$= \frac{\mathbb{P}(N(4) < 1)\ \mathbb{P}(N(5) < 3 \mid N(4) < 1)}{\mathbb{P}(N(5) < 3)}$$

$$= \frac{\mathbb{P}(N(4) = 0)\ \mathbb{P}(N(5) - N(4) < 3 \mid N(4) = 0)}{\mathbb{P}(N(5) < 3)}$$

$$= \frac{\mathbb{P}(N(4) = 0)}{\mathbb{P}(N(5) < 3)}\ \mathbb{P}(N(5) - N(4) < 3)$$ (since independent intervals)

• And recall $N(t) - N(s) \sim \text{Poisson}((t-s)\lambda)$

• So $\mathbb{P}(N(4) = 0) = e^{-4\lambda}$

$$\mathbb{P}(N(5) < 3) = e^{-5\lambda} + (5\lambda) e^{-5\lambda} + \frac{25\lambda^2}{2} e^{-5\lambda}$$
$$= e^{-5\lambda}\left(1 + 5\lambda + \frac{25}{2}\lambda^2\right)$$

$$\mathbb{P}(N(5) - N(4) < 3) = e^{-\lambda} + \lambda e^{-\lambda} + \frac{\lambda^2}{2} e^{-\lambda}$$
$$= e^{-\lambda}\left(1 + \lambda + \frac{\lambda^2}{2}\right)$$

• Thus $\mathbb{P}(T_1 > 4 \mid T_3 > 5) = \dfrac{\cancel{e^{-5\lambda}}\left(1 + \lambda + \frac{\lambda^2}{2}\right)}{\cancel{e^{-5\lambda}}\left(1 + 5\lambda + \frac{25}{2}\lambda^2\right)}$

$$= \boxed{\frac{1 + \lambda + \frac{\lambda^2}{2}}{1 + 5\lambda + \frac{25}{2}\lambda^2}}$$

2c. • Recall that $\mathrm{Cov}(X,Y)=0$ when X,Y independent RVs

• Also, $\mathrm{Cov}(X+Y,Z) = \mathrm{Cov}(X,Z) + \mathrm{Cov}(Y,Z)$ for all X,Y,Z RVs.

• So
$$\begin{aligned}\mathrm{Cov}(N_6, N_4-N_1) &= \mathrm{Cov}(N_6-N_4, N_4-N_1) \\ &+ \mathrm{Cov}(N_4-N_1, N_4-N_1) \\ &+ \mathrm{Cov}(N_1-N_0, N_4-N_1)\end{aligned} \quad (\star)$$

• Note $\mathrm{Cov}(N_6-N_4, N_4-N_1) = 0$
and $\mathrm{Cov}(N_1-N_0, N_4-N_1) = 0$ (independent increments)

• So
$$\begin{aligned}(\star) &= \mathrm{Cov}(N_4-N_1, N_4-N_1) \\ &= \mathrm{Var}(N_4-N_1) \\ &= \boxed{3\lambda}\end{aligned}$$
(since $N_4-N_1 \sim \mathrm{Pois}(3\lambda)$)

2 Question 2 **10 / 10**

Part a) : 4 points

✓ **- 0 pts** **Everything correct**

- 1 pts Minor mistakes

- 0 pts There are some issues

Part b) : 3 points

✓ **- 0 pts** **Everything correct**

- 1.5 pts Incorrect way of splitting the conditional probability.

- 2 pts Incorrect reasoning

- 0 pts There are some issues

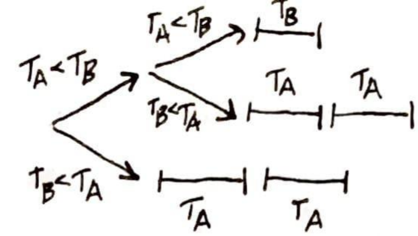
Part c) : 3 points

✓ **- 0 pts** **Everything correct**

- 1 pts Minor mistakes

- 2 pts Incorrect reasoning regarding the independent increments condition.

- 0 pts There are some issues



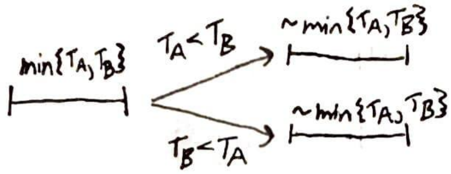
3a. • Let T_A, T_B be the times for computer 1 & 2 to mine a block resp.
• We know $T_A \sim Exp(\lambda_A)$ & $T_B \sim Exp(\lambda_B)$, $\lambda_A = 1/5$ $\lambda_B = 1/10$

• Let T = [time until all three blocks are mined]
given the blocks are assigned as in (a)

• Note
$$\mathbb{E}(T) = \mathbb{P}(T_B<T_A)\big(\mathbb{E}(T_A)+\mathbb{E}(T_A)\big) + \mathbb{P}(T_A<T_B)\Big(\mathbb{P}(T_A<T_B)\,\mathbb{E}(T_B) + \mathbb{P}(T_B<T_A)\big(\mathbb{E}(T_A)+\mathbb{E}(T_A)\big)\Big)$$
(by memoryless property, using the diagram above)
$$= \tfrac{1}{3}(2\cdot 5) + \tfrac{2}{3}\left(\tfrac{2}{3}(10) + \tfrac{1}{3}(2\cdot 5)\right)$$
$$= \boxed{10}$$

• Since $\mathbb{P}(T_A<T_B) = \dfrac{\lambda_A}{\lambda_A+\lambda_B} = \dfrac{2}{3}$

and $\mathbb{E}(T_A) = \dfrac{1}{\lambda_A} = 5$
$\mathbb{E}(T_B) = \dfrac{1}{\lambda_B} = 10$



36. · Let T = [time until all three blocks are mined] given the blocks are assigned as described in (b)

· Note
$$\mathbb{E}(T) = \mathbb{E}(\min\{T_A, T_B\}) + \mathbb{P}(T_A < T_B)\,\mathbb{E}(\min\{T_A, T_B\}) + \mathbb{P}(T_B < T_A)\,\mathbb{E}(\min\{T_A, T_B\})$$

(by memoryless property)

$$= \left(\frac{1}{\lambda_A + \lambda_B}\right) + \left(\frac{\lambda_A}{\lambda_A + \lambda_B}\right)\left(\frac{1}{\lambda_A + \lambda_B}\right) + \left(\frac{\lambda_B}{\lambda_A + \lambda_B}\right)\left(\frac{1}{\lambda_A + \lambda_B}\right)$$

(since $\min\{T_A, T_B\} \sim Exp(\lambda_A + \lambda_B)$)

$$= \left(\frac{10}{3}\right)(1+1)$$

$$= \boxed{\frac{20}{3}}$$

3 Question 3 **5 / 10**

Basic knowledge

✓ **- 0 pts** **The solution is showing some understanding on the exponential race.**

- 3 pts No sign of understanding on the exponential race

Understanding of the problem

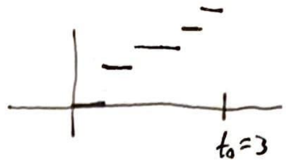
- 0 pts Perfect understanding and computation

- 1 pts Almost perfect understanding with only minor mistakes

- 3 pts Decent understanding with moderate mistakes

✓ **- 5 pts** **Largely inaccurate understanding with major mistakes**

- 7 pts Essentially no understanding is provided.



4a. • Let $\{N(t)\}_{t\geq 0}$ the poisson process for the spectators, where $t=$ [time past 5PM]
so $N \sim PP(\lambda=16)$

• Let $t_0=3$, the time the concert opens

• Note $W=\sum_{i=1}^{X}(t_0-T_i)$ and $X=N(t_0)$

• So $\mathbb{E}(W|X=k) = \mathbb{E}\left(\sum_{i=1}^{k}(t_0-T_i) \,\middle|\, N(t_0)=k\right)$

$$= kt_0 - \sum_{i=1}^{k}\mathbb{E}(T_i|N(t_0)=k) \qquad \forall k\in\mathbb{N}_0$$

• We must compute $\underset{\sim\sim\sim\sim}{\mathbb{E}(T_i/N(t_0)=k)}$ $\quad \forall k\in\mathbb{N}_0 \quad \forall i\in\{1,2,\dots k\}$

• Let $s\in[0,t_0]$. Let i

• Note $\mathbb{P}(T_i>s\,|N(t_0)=k) = \mathbb{P}(N(s)<i\,|\,N(t_0)=k)$

$$= \sum_{j=0}^{i-1}\mathbb{P}(N(s)=j\,|N(t_0)=k)$$

$$= \sum_{j=0}^{i-1}\binom{k}{j}\left(\frac{s}{t_0}\right)^j\left(1-\frac{s}{t_0}\right)^{k-j} \qquad \text{(by conditioning)}$$

• So $\mathbb{E}(T_i|N(t_0)=k) = \int_0^{t_0}\mathbb{P}(T_i>s\,|N(t_0)=k)\,ds$

$$= \sum_{j=0}^{i-1}\binom{k}{j}\int_0^{t_0}\left(\frac{s}{t_0}\right)^j\left(1-\frac{s}{t_0}\right)^{k-j}ds$$

$$= \sum_{j=0}^{i-1}\binom{k}{j}t_0\int_0^1 v^j(1-v)^{k-j}ds \qquad (v=s/t_0)$$

$$= \sum_{j=0}^{i-1}\binom{k}{j}t_0\left(\frac{j!(k-j)!}{(k+1)!}\right)$$

$$= \frac{t_0\, i}{k+1}$$

4a (cont.) · Then $E(W|X=k) = kt_0 - \sum_{i=1}^{k} \left(\frac{t_0 i}{k+1}\right)$

$$= kt_0 - \frac{t_0}{\cancel{k+1}}\left(\frac{\cancel{(k+1)}(k)}{2}\right)$$

$$= kt_0 - \frac{1}{2}kt_0$$

$$= \boxed{\frac{1}{2}kt_0}$$

$$= \boxed{\frac{3}{2}k}$$

46 $\mathbb{E}(W) = \sum_{k=0}^{\infty} \mathbb{E}(W \mid N(t_0)=k)\, \mathbb{P}(N(t_0)=k)$

$$= \sum_{k=0}^{\infty} \left(\tfrac{1}{2} k t_0\right) \left(\frac{(\lambda t_0)^k}{k!} e^{-\lambda t_0}\right) \qquad \text{(since } N(t_0)-N(0) \sim \text{Poisson}(\lambda t_0)\text{)}$$

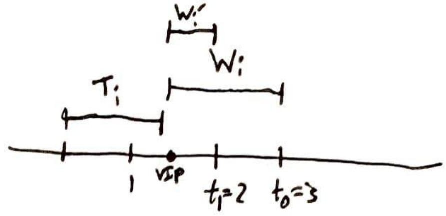
$$= \tfrac{1}{2} t_0 (\lambda t_0) \left(\sum_{k=1}^{\infty} \frac{(\lambda t_0)^{k-1}}{(k-1)!}\right) e^{-\lambda t_0}$$

$$= \tfrac{1}{2} t_0^2 \lambda\, \cancel{e^{\lambda t_0} e^{-\lambda t_0}}$$

$$= \boxed{\tfrac{1}{2} t_0^2 \lambda}$$

$$= \tfrac{1}{2} \cdot 9 \cdot 16$$

$$= \boxed{72}$$



- Let $t_1 = 2$. Let $p = 0.15$, $q = 1-p$

4c. • Let W_i be the wait time for the ith spectator, (in the original scenario)
so $W_i = t_0 - T_i$ and $W = \sum_{i=1}^{X} W_i$

- Let $Y_i = \begin{cases} 1, & i \text{ is VIP} \\ 0, & \text{else} \end{cases}$

- Let W_i' be the new wait time for the ith spectator, with VIP rules put in place.
- Let W' be the new total wait time, $W' = \sum_{i=1}^{X} W_i'$
- Note $W_i' = \begin{cases} W_i, & Y_i = 0 \\ W_i - 1 & Y_i = 1 \text{ and } T_i < t_1 \\ 0 & Y_i = 1 \text{ and } T_i \geq t_1 \end{cases}$

- Let $i \in \{1, 2, \dots X\}$
- Note $\mathbb{E}(W_i') = \mathbb{P}(Y_i = 0)\,\mathbb{E}(W_i) + \mathbb{P}(Y_i = 1)\left(\mathbb{P}(T_i > t_1) \cdot 0 + \mathbb{P}(T_i < t_1)(\mathbb{E}(W_i) - 1)\right)$

$$= q\,\mathbb{E}(W_i) + p\left(\tfrac{1}{3} \cdot 0 + \tfrac{2}{3}(\mathbb{E}(W_i) - 1)\right)$$

(since $i \leq X$ implies W_i is uniform on $[0, t_0]$)

$$= q\,\mathbb{E}(W_i) + \tfrac{2}{3} p\,(\mathbb{E}(W_i) - 1)$$

- So $\mathbb{E}(W') = \mathbb{E}\left(\sum_{i=1}^{X} W_i'\right)$

$$= q\,\mathbb{E}(W) + \tfrac{2}{3} p\,(\mathbb{E}(W) - 1)$$

$$= \boxed{q \tfrac{1}{2} t_0^2 \lambda + \tfrac{2}{3} p\,(t_0^2 \lambda \tfrac{1}{2} - 1)}$$

$$= \boxed{\tfrac{1}{2} 3^2 (0.85) \cdot 16 + \tfrac{2}{3}(0.15)(3^2 \cdot 16 \cdot \tfrac{1}{2} - 1)}$$

4 Question 4 **8.8 / 10**

Part a) : 5 points

✓ **- 0 pts** **Everything correct**

- 2 pts Moderate mistakes

- 3.5 pts Critical misunderstanding on the conditional distribution of waiting times given $$X = k$$

- 5 pts Essentially no work is provided

Part b) : 2.5 points

- 0 pts Everything correct (ignoring errors carried forward)

✓ **- 1.2 pts** **There are some issues**

- 2.5 pts Essentially no work is provided

Part c) : 2.5 points

✓ **- 0 pts** **Everything correct (ignoring errors carried forward)**

- 1.2 pts There are some issues

- 2.5 pts Essentially work is provided

1 Unit mismatch. ($$t_0=3$$ hours and $$\lambda=16$$ people per minute)

2 Unit mismatch, but I did not mark off points for errors carried forward.

· Let $p = 3/5$ $q = 2/5$

5a. · Let $N \sim PP(\lambda)$ be the poisson process of burnouts,
so $N(t) = [\#$ of bulbs burnt out by time $t]$ $\forall t \geq 0$
where $\lambda = 1/2$

· Let $\{T_i\}_{i \in \mathbb{N}}$ be the arrival times of N

· Let $Y_k = \begin{cases} 1 & \text{burnout } k \text{ was type-1} \\ 2 & \text{burnout } k \text{ was type-2} \end{cases}$ $\forall k \in \mathbb{N}$

· Note $p = \mathbb{P}(Y_k = 1)$ $\forall k \in \mathbb{N}$

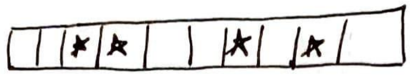
· Note $\{Y_k\}_{k \geq 1}$ is a sequence of IID RVs independent of N.

· Thus by Thinning the processes $N_i(t) = [\#$ burnouts of type i up to time $t]$ $i \in \{1, 2\}$
are Poisson processes with $N_1 \sim PP(\lambda p)$ $N_2 \sim PP(\lambda q)$
and are independent.

· Let $\{R_i\}_{i \in \mathbb{N}}$ $\{S_i\}_{i \in \mathbb{N}}$ be the arrival times of N_1 & N_2 respectively

· Let $t \in [0, \infty)$ be such that no bulbs have burnt out yet,
so $N(t) = 0$. Then $N_1(t) = N_2(t) = 0$

· So
$$
\begin{aligned}
\mathbb{E}(R_1 \mid N_1(t) = 0) &= \mathbb{E}((R_1 - t) + t \mid N_1(t) = 0) \\
&= \mathbb{E}(R_1 - t \mid N_1(t) = 0) + t \\
&= \mathbb{E}(R_1) + t \qquad \text{(lack of memory prop. of } N_1\text{)} \\
&= \boxed{\frac{1}{\lambda p} + t} \qquad \text{(since } R_1 \sim Exp(\lambda p)\text{)}.
\end{aligned}
$$



56. • Let $n \in \mathbb{N}$ and $k \in \{1, 2, \dots n\}$

• Consider $\mathbb{P}(N_2(t) = k \mid N(t) = n)$

• Note since each light bulb type Y_i is selected independently from everything else this is just

$$\mathbb{P}(N_2(t) = k \mid N(t) = n) = \mathbb{P}_{X \sim Binom(n, q)}(X = k)$$

$$= \binom{n}{k} q^k p^{n-k}$$

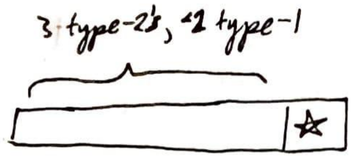
• So $\mathbb{P}(\{\text{exactly four type-2 bulbs failed up to the time of } 10^{th} \text{ failure}\})$

$$= \mathbb{P}(N_2(t) = 4 \mid N(t) = 10)$$

$$= \binom{10}{4} q^4 p^6$$

$$= \boxed{\binom{10}{4} \left(\frac{2}{5}\right)^4 \left(\frac{3}{5}\right)^6}$$

$$\approx \boxed{0.251}$$



5c.

• $\mathbb{P}$("four type-2's fail before two type-1's fail") $= \mathbb{P}(S_4 < R_2)$

$$= \sum_{k=0}^{1} \mathbb{P}(\text{"}k \text{ type-1's fail and } 4 \text{ type-2's fail, ending in type-2"})$$

$$= \sum_{k=0}^{1} \binom{k+3}{k} p^k q^{3+1}$$

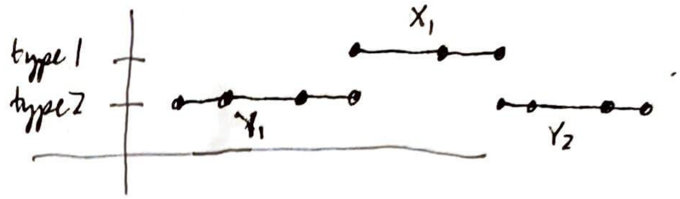
$$= \binom{3}{0} q^4 + \binom{4}{1} p q^4$$

$$= 1q^4 + 4pq^4$$

$$= q^4(1+4p)$$

$$= \boxed{\left(\tfrac{2}{5}\right)^4 \left(1+4\left(\tfrac{3}{5}\right)\right)}$$

$$= \boxed{0.087}$$



5d. • We can view this as an alternating renewal process

• Let X_n = [duration of nth chain of type-1s]
and Y_n = [duration of nth chain of type-2s]

• Let P_n, Q_n be the lengths of the nth chains of type-1 and type-2 respectively

• Let $\{x_n\}, \{y_n\}$ be the interarrival times of N_1 and N_2 resp. (from part (a))

• Note $X_n = \sum_{i=k}^{P_n} x_i$ where k is s.t. R_k is first burnout of chain X_n

• Note $P_n \sim$ Geometric(q) (since the chain ends once a type- is used)
• And $x_n \sim Exp(\lambda)$
• So from lecture we know $X_n \sim Exp(q\lambda)$

• Thus $\mathbb{E}(X_n) = \frac{1}{p\lambda}$

• Similar logic shows $\mathbb{E}(Y_n) = \frac{1}{q\lambda}$

• Thus, we know
$\lim_{t\to\infty} \mathbb{P}$("type-2 used at time t")
= [limiting fraction of time in type-2 bulb]
$= \frac{1/p\lambda}{\frac{1}{q\lambda} + 1/p\lambda} = \boxed{\frac{q}{p+q}}$

5 Question 5 **10 / 10**

Part a) : 2 points

✓ **- 0 pts** **Everything correct**

- 1 pts Minor mistakes (such as missing the term $$t$$, computation mistakes)

- 0 pts There are some issues

Part b) : 3 points

✓ **- 0 pts** **Everything correct**

- 0 pts There are some issues

Part c) : 3 points

✓ **- 0 pts** **Everything correct**

- 1 pts Minor mistakes

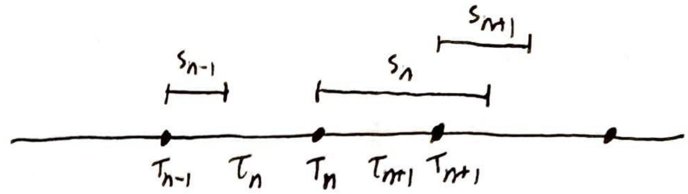
- 0 pts There are some issues

Part d) : 2 points

✓ **- 0 pts** **Everything correct**

- 1 pts Mistakes in applying SLLN

- 0 pts There are some issues



6a. • Let $\{N(t)\}_{t \geq 0}$ be the poisson process of Tourist arrivals, that is $N(t) = [\#$ of Tourists arrived by $t]$ and we know $N \sim PP(\lambda)$

• Let $\{T_i\}_{i \in \mathbb{N}}$ be the arrival times for N, and let $\{\tau_i\}_{i \in \mathbb{N}}$ be the interarrival times for N

• We know $\tau_i \sim Exp(\lambda)$

• Let $\{S_i\}_{i \in \mathbb{N}}$ be the time needed for the ith tourist to complete tour

• We know $S_i \sim Exp(\lambda_c)$ where $\lambda_c = \boxed{\frac{1}{10}}$

• Let $W_i = \begin{cases} 1, & \text{ith tourist completes} \\ 0, & \text{else} \end{cases}$ $\forall i \in \mathbb{N}$

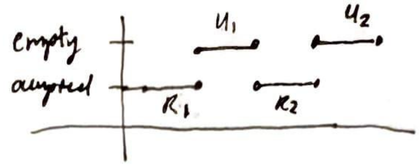
• Note $\mathbb{P}(\text{"ith tourist completes"}) = \mathbb{P}(\tau_{n+1} > S_n)$

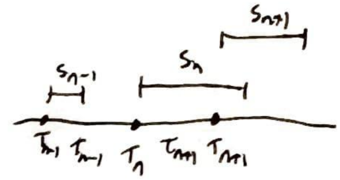
$$= \frac{\lambda_c}{\lambda + \lambda_c} \quad \text{(minimum of exp dist index)}$$

$$= \frac{1/10}{\lambda + 1/10}$$

• Note this doesn't depend on n, so

$$[\text{fraction of tourists that complete}] = \boxed{\frac{1/10}{\lambda + 1/10}}$$





66. • We can view this as an alternating renewal process

• Let R_n = [time that nth tourist occupies the spot]
and U_n = [time that spot is empty, after nth tourist, before (n+1)th tourist]
• Note $R_n = \min\{T_{n+1}, S_n\}$, so $R_n \sim Exp(\lambda + \lambda_c)$
• Also, $U_n = \max\{0, T_{n+1} - S_n\}$

• Then $\mathbb{E}(R_n) = \dfrac{1}{\lambda + \lambda_c}$

(?) • And $\mathbb{E}(U_n) = \mathbb{P}(S_n < T_{n+1})\, \mathbb{E}(T_{n+1} - S_n \mid S_n < T_{n+1})$
$+ \mathbb{P}(S_n > T_{n+1}) \cdot (0)$

$= \mathbb{P}(S_n < T_{n+1})\, \mathbb{E}(T_{n+1})$ (by memoryless property)

$= \left(\dfrac{\lambda_c}{\lambda + \lambda_c}\right)\left(\dfrac{1}{\lambda}\right)$

$= \left(\dfrac{1/10}{1/10 + \lambda}\right)\left(\dfrac{1}{\lambda}\right)$

• So [fraction of time spent unoccupied] $= \dfrac{\mathbb{E}(U_1)}{\mathbb{E}(R_1) + \mathbb{E}(U_1)}$

$= \dfrac{\lambda_c/\lambda}{1 + \lambda_c/\lambda}$

$= \dfrac{1}{\lambda/\lambda_c + 1}$

$= \boxed{\dfrac{1}{10\lambda + 1}}$

6c. • Let k be the Tourist index of Hadvar

•

• Note $\mathbb{P}(\{\text{Hadvar completes}\}) = \mathbb{P}(S_k < T_{k+1} + T_{k+2} + T_{k+3})$

$$= 1 - \mathbb{P}(S_k > T_{k+1} + T_{k+2} + T_{k+3})$$

• Note $\mathbb{P}(S_k > T_{k+1} + T_{k+2} + T_{k+3}) = \mathbb{P}(S_k > T_{k+1})\, \mathbb{P}(S_k > T_{k+1} + T_{k+2} + T_{k+3} \mid S_k > T_{k+1})$

$$+ \mathbb{P}(S_k < T_{k+1}) \cdot 0$$

$$= \mathbb{P}(S_k > T_{k+1})\, \mathbb{P}(S_k > T_{k+2} + T_{k+3})$$

(memoryless property)

• Similar logic shows $\mathbb{P}(S_k > T_{k+2} + T_{k+3}) = \mathbb{P}(S_k > T_{k+2})\, \mathbb{P}(S_k > T_{k+3})$

• So $\mathbb{P}(\{\text{Hadvar completes}\}) = 1 - \left(\dfrac{\lambda_c}{\lambda + \lambda_c}\right)^3$

$$= \boxed{1 - \left(\frac{1/10}{1/10 + \lambda}\right)^3}$$

6 Question 6 **8 / 10**

Part a) : 4 points

- **- 0 pts** Everything correct
- ✓ **- 1 pts Minor issues**
- **- 2 pts** Major issues
- **- 4 pts** Essentially no work is provided

Part b) : 3 points

- ✓ **- 0 pts Everything correct**
- **- 1 pts** Minor issues
- **- 2 pts** Major issues
- **- 3 pts** Essentially no work is provided

Part c) : 3 points

- **- 0 pts** Everything correct
- ✓ **- 1 pts Minor issues**
- **- 2 pts** Major issues
- **- 3 pts** Essentially no work is provided

3 10 minutes = 1/6 hours

4 Error carried forward

5 You should have $$\lambda$$

- Let $\lambda = 5$, and $N \sim PP(\lambda)$ be the student arrival poisson process, with arrival times $\{T_i\}_{i \in \mathbb{N}}$

7a. - This can be viewed as a Alternating Renewal Process

- Let S_n = [duration nth student who visited stayed]

and U_n = [duration between $(n-1)^{th}$ students exit and nth student's entrance]

- Note that S_n's and U_n's are all independent (for U_n's this is because of memorylessness prop. of N)

- Note $U_n \sim Exp(\lambda)\ \forall n$ (by memoryless property of Poisson process of students arrival)

- And $S_n \sim Unif(6, 30)\ \forall n$ (given)

- So [limiting fraction of time at which students are in Baech's office]

$$= \frac{\mathbb{E}(S_1)}{\mathbb{E}(S_1) + \mathbb{E}(U_1)}$$

$$= \frac{36/2}{36/2 + 1/\lambda}$$

$$= \boxed{\frac{18}{18 + 1/5}}$$

$$\approx \boxed{0.989}$$

76. • Let $\{N'(t)\}_{t\geq 0}$ be the renewal process of students that stay

• Denote N and N''s interarrival times as $\{\tau_i\}_{i\in\mathbb{N}}$ $\{\tau_i'\}_{i\in\mathbb{N}}$ respectively
• Note $\tau_n \sim \text{Exp}(\lambda)$ $\forall n$ and $\tau_n' = U_n + S_n$ $\forall n$

• Note
$$\lim_{t\to\infty}\left(\frac{N(t)}{t}\right) = \frac{1}{\mathbb{E}(\tau_1)} \quad (\text{SLLN})$$
$$= \lambda$$

•

• And
$$\lim_{t\to\infty}\left(\frac{N'(t)}{t}\right) = \frac{1}{\mathbb{E}(\tau_1')}$$
$$= \frac{1}{\mathbb{E}(U_n)+\mathbb{E}(S_n)}$$
$$= \frac{1}{1/\lambda + 18}$$

• So
$$\lim_{t\to\infty}\left(\frac{N'(t)}{N(t)}\right) = \lim_{t\to\infty}\left(\frac{N'(t)/t}{N(t)/t}\right) = \boxed{\frac{1}{1+18\lambda}}$$
$$= \frac{1}{1+18/5}$$
$$= \frac{5}{5+18}$$
$$= \boxed{\frac{5}{23}}$$

7 Question 7 **10 / 10**

Part a) : 5 points

✓ **- 0 pts Everything correct**

- **- 1 pts** Minor issues
- **- 3 pts** Major issues
- **- 5 pts** Essentially no work is provided

Part b) : 5 points

✓ **- 0 pts Everything correct (ignoring errors carried forward)**

- **- 1 pts** Minor issues
- **- 3 pts** Major issues
- **- 5 pts** There are some issues

6 $$\lambda$$ = 5 per hour = $$\frac{1}{2}$$ per minute

8. • We need to show $\tilde{N}$ satisfies the definition of a non-homogeneous PP.

• Note $\tilde{N}(0) = N(\mu(0)) = N(\int_0^0 \lambda(s)ds) = N(0) = 0$ (since $N \sim PP(1)$)

• Next, note for any $r, t \in [0,\infty)$

$$\tilde{N}(t) - \tilde{N}(r) = N(\mu(t)) - N(\mu(r))$$
$$\sim \text{Poisson}(1 \cdot (\mu(t) - \mu(r))) \quad \text{(since } N \sim PP(1))$$
$$= \text{Poisson}(\textstyle\int_0^t \lambda(s)ds - \int_0^r \lambda(s)ds)$$
$$= \text{Poisson}(\textstyle\int_r^t \lambda(s)ds)$$

• Lastly note for any $0 \le t_0 \le t_1 \le t_2 \le t_3$, $0 \le \mu(t_0) \le \mu(t_1) \le \mu(t_2) \le \mu(t_3)$ since $\mu(t) = \int_0^t \lambda(s)ds$ and $\lambda(s) \ge 0 \ \forall s \in [0,\infty)$. Thus

$$\tilde{N}(t_3) - \tilde{N}(t_2) = N(\mu(t_3)) - N(\mu(t_2))$$

and $\tilde{N}(t_1) - \tilde{N}(t_0) = N(\mu(t_1)) - N(\mu(t_0))$

are <u>independent</u> (since $N \sim PP(1)$)

• Therefore $\tilde{N}$ is a poisson process w/ rate function $\lambda(\cdot)$

8 Question 8 10 / 10

✓ **- 0 pts** **Everything correct**

Solution based on the definition

- 1 pts $$\tilde{N}(0) = 0$$ is not checked

- 3.5 pts Poisson increment condition is not rigorously checked (i.e., the solution is not properly and explicitly demonstrating understanding on this condition.)

- 2.5 pts Independent increments condition is not rigorously checked (i.e., the solution is not properly and explicitly demonstrating understanding on this condition.)

- 0 pts There are some issues

- 10 pts Essentially no work is provided

9a. • For convenience, denote $f_k(t,h) = \mathbb{P}(N(t+h)-N(t)=k)$

• Note then $\lim_{h\to 0^+}\left(\frac{f_1(t,h)}{h}\right) = \lim_{h\to 0^+}\left(\frac{\mathbb{P}(N(t+h)-N(t)=1)}{h}\right)$

$= \lambda$

• And $\lim_{h\to 0^+}\left(\frac{\sum_{k=2}^{\infty} f_k(t,h)}{h}\right) = \lim_{h\to 0^+}\left(\frac{\mathbb{P}(N(t+h)-N(t)\geq 2)}{h}\right)$ (A)

$= 0$

• Also $\lim_{h\to 0^+}\left(\frac{f_0(t,h)-1}{h}\right) = \lim_{h\to 0^+}\left(\frac{f_0(t,h)-\sum_{k=0}^{\infty} f_k(t,h)}{h}\right)$

$= \lim_{h\to 0^+}\left(\frac{-f_1(t,h)}{h} - \frac{\sum_{k=2}^{\infty} f_k(t,h)}{h}\right)$

$= -\lambda - 0$

$= -\lambda$

• Also, since N is a counting process, we know

$\mathbb{P}(N(t+h)=k) = \sum_{i=0}^{k} \mathbb{P}(N(t+h)=k \mid N(t)=i)\,\mathbb{P}(N(t)=i)$ (total prob. thm)

$= \sum_{i=0}^{k} \mathbb{P}(N(t+h)-N(t)=k-i \mid N(t)=i)\,\mathbb{P}(N(t)=i)$

$= \sum_{i=0}^{k} f_{k-i}(t,h)\, g_i(t)$

9a. (cont.)

- $$g_k'(t) = \lim_{h \to 0^+} \left(\frac{g_k(t+h) - g_k(t)}{h} \right)$$
$$= \lim_{h \to 0^+} \left(\frac{\sum_{i=0}^{k} f_{k-i}(t,h) g_i(t) - g_k(t)}{h} \right)$$

- When $k=0$, $$g_0'(t) = \lim_{h \to 0^+} \left(\frac{(f_0(t,h)-1)\, g_0(t)}{h} \right)$$
$$= g_0(t) \cdot (-\lambda)$$
$$= \boxed{-\lambda g_0(t)}$$

- When $k \in \mathbb{N}$, $$g_0'(t) = \lim_{h \to 0^+} \left(\frac{\sum f_{k-i}(t,h) g_i(t) + \sum f_{k-i}(t,h) g_i(t) - g_k(t)}{h} \right)$$
$$= \lim_{h \to 0^+} \left(\frac{\sum_{i=k-1}^{k} (f_{k-i}(t,h)\, g_i(t)) - g_k(t)}{h} \right) + \lim_{h \to 0^+} \left(\frac{\sum_{i=0}^{k-2} f_{k-i}(t,h)}{h} \right)$$
$$= \lim_{h \to 0^+} \left(\frac{(f_0(t,h)-1)\, g_k(t) + f_1(t,h)\, g_{k-1}(t)}{h} \right) + 0 \quad \text{(because of ★)}$$
$$= (-\lambda)\, g_k(t) + (\lambda) g_{k-1}(t)$$
$$= \boxed{-\lambda g_k(t) + \lambda g_{k-1}(t)}$$

By product rule,

96. $\frac{d}{dt}(e^{\lambda t} g_k(t)) = \lambda e^{\lambda t} g_k(t) + e^{\lambda t} g_k'(t)$

$$= e^{\lambda t}(\lambda g_k(t) + g_k'(t))$$

$$= \begin{cases} e^{\lambda t}(0) = 0 & k=0 \\ e^{\lambda t}(\lambda g_{k-1}(t)) & k \geq 1 \end{cases}$$

- We can use these differential equations to find $g_k(t)$ $k \in \mathbb{N}_0$
- Integrating both sides of $\frac{d}{dt}(e^{\lambda t} g_0(t)) = 0$,

$$e^{\lambda t} g_0(t) = c \text{ where } c \in \mathbb{R}$$

- Then plugging in $t=0$, $e^{\lambda \cdot 0} g_0(0) = c$
  $\Rightarrow g_0(0) = c$
- So $g_0(0) = \mathbb{P}(N(0)=0) = 1 = c$
- Then $\boxed{g_0(t) = c/e^{\lambda t} = e^{-\lambda t}}$
- Now, integrating both sides of $\frac{d}{dt}(e^{\lambda t} g_1(t)) = \lambda e^{\lambda t} g_0(t) = \lambda e^{\lambda t}(e^{-\lambda t}) = \lambda$

$$e^{\lambda t} g_1(t) = \int \lambda \, dt = \lambda t + c, \quad c \in \mathbb{R}$$

- Then $1 \cdot g_1(0) = \lambda \cdot 0 + c \Rightarrow g_1(0) = \mathbb{P}(N(0)=1) = 0 = c$
- So $\boxed{g_1(t) = \lambda t e^{-\lambda t}}$

9 Question 9 **10 / 10**

Verifying $$g_0'(t)=-\lambda g_0(t)$$: 6 points

✓ **- 0 pts** **Everything correct**

- 1 pts Minor issues

- 2.5 pts Moderate issues

- 4 pts Major issues

- 6 pts Essentially no work is provided

Verifying $$g_0(t)=e^{-\lambda t}$$: 4 points

✓ **- 0 pts** **Everything correct**

- 1 pts Minor issues

- 1.5 pts Verified $$(e^{\lambda t}g_0(t))' = 0$$ but did not explain why this leads to the desired solution

- 4 pts Essentially no work is provided