22S-MATH-170E-LEC-1 Midterm 2

[redacted]

TOTAL POINTS

42 / 50

QUESTION 1

1 Problem 1 **16 / 16**

✓ **- 0 pts** **Correct**

- 1 pts (a) error

- 3 pts (a) multiple critical errors

- 4 pts (a) little progress towards correct answer

- 1 pts (b) error

- 1 pts (b) left infinite sum in answer

- 2 pts Only computed (correctly) half of the conditional probability fraction

- 3 pts (b) multiple critical errors--or very incomplete

- 4 pts (b) little progress towards correct answer

- 1 pts (c) error

- 1 pts (c) left infinite sum in answer

- 3 pts (c) multiple critical errors

- 4 pts (c) little progress

- 1 pts (d) error

- 2 pts (d) Mostly right, but incomplete

- 3 pts (d) multiple critical errors

- 4 pts (d) little progress

QUESTION 2

2 Problem 2 **10 / 10**

✓ **- 0 pts** **Correct**

- 1 pts (a) gives a definition that is only valid for discrete or continuous

- 3 pts (a) describes properties of the mgf, but does not give the definition

- 4 pts (a) no progress

- 1 pts (b) minor error

- 2 pts (b) incorrect/missing $$E[X^2]$$

- 2 pts (b) incorrect/missing $$E[X]$$

- 2 pts (b) incorrect/missing formula to compute variance

QUESTION 3

3 Problem 3 **12 / 12**

✓ **- 0 pts** **Correct**

- 2 pts a) did not specify the different cases of the PDF correctly

Nice!

QUESTION 4

4 Problem 4 **4 / 12**

- 0 pts Correct

- 2 pts a) Missed the atom at w=0

- 8 **Point adjustment**

a) 2/7 some work, answer far from correct
b) 2/5 this is the correct answer, but is not correct given your answer to part a). The CDF you've given in a) produces E(W) = 1/2.

University of California, Los Angeles
Spring 2022

Instructor: T. Arant
Date: May 18, 2022

Name: [redacted] UCLA ID: [redacted] Signature: [redacted]

MATH 170E: PROBABILITY
MIDTERM 2

This exam contains 8 pages (including this cover page) and 4 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

This is a closed book exam. No cheat sheets, calculators, phones, laptops, etc. are allowed.

You are required to show your work on each problem of this exam. The following rules apply:

- **All answers must be justified. Mysterious or unsupported answers will not receive credit.** A correct answer, unsupported by calculations and/or explanation will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- **If you use a theorem or proposition from class or the notes or the textbook or a result established in the homework, you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- When applicable, it is acceptable (and even preferable) for your final answers to contain unsimplified terms of the form n^m, $n!$, ${}_nP_r$ and ${}_nC_r$.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	16	
2	10	
3	12	
4	12	
Total:	50	

Good luck!

Formula sheet.

- The mgf of $X \sim \text{Binom}(n,p)$ is $M(t) = ((1-p) + pe^t)^n$.
- The mgf of $X \sim \text{Geo}(p)$ is

$$M(t) = \frac{pe^t}{1-(1-p)e^t}, \qquad t < -\ln(1-p).$$

- The mgf of $X \sim \text{Pois}(\lambda)$ is $M(t) = \exp[\lambda(e^t - 1)]$.
- The mgf of a $X \sim \text{Unif}[a,b]$ is

$$M(t) = \begin{cases} \frac{e^{bt}-e^{at}}{t(b-a)} & \text{if } t \neq 0, \\ 1 & \text{if } t = 0. \end{cases}$$

- The mgf of $X \sim \text{Exp}(\theta)$ is

$$M(t) = \frac{1}{1-\theta t}, \qquad t < \frac{1}{\theta}.$$

- For $X \sim \Gamma(\alpha, \theta)$, the pdf is

$$f(x) = \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-x/\theta}, \qquad t \geq 0,$$

$E[X] = \alpha\theta$, $\text{Var}(X) = \alpha\theta^2$.

- The $\text{Exp}(\theta)$ distribution is the same as $\Gamma(1, \theta)$.
- The $\chi^2(r)$ distribution is the same as $\Gamma(r/2, 2)$.
- For $X \sim N(\mu, \sigma^2)$, the pdf is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}},$$

and the mgf is $M(t) = \exp(\mu t + \sigma^2 t^2/2)$.

TG

1. Customers arrive at a store according to a Poisson process with rate $\lambda = 2$ per hour.

   At the start of each hour, a new clerk starts an hour-long shift running the store. At the end of the shift, the clerk is replaced by a new clerk.

   If 2 or more customers arrive at the store during a shift, then the clerk for that shift gets a bonus.

   For each question below, do **not** leave an infinite sum in your answer.

   (a) (4 points) What is the probability that the first customer of the day arrives some time during the first three shifts?

$X \sim \text{Pois}(2 \cdot 3) = \text{Pois}(6)$ X = Number of arrivals over 3 shifts

~~Y = First customer arriving in first 3 shifts~~

~~$Y \sim \text{Exp}(\frac{1}{6})$~~

$\lambda = 6$

$\theta = \frac{1}{\lambda}$

$P(\text{First arrives in first 3 shifts}) = 1 - P(\text{None arrive in first 3 shifts})$

$= 1 - P(X=0)$

$= 1 - \frac{6^0 e^{-6}}{0!} = 1 - e^{-6}$

   (b) (4 points) Given that at least one customer arrived during the first shift, what is the conditional probability that the clerk working the first shift received a bonus?

$\lambda = 2/\text{hr}$

$\lambda t = 2$

X = Number of customers in first shift $\sim \text{Pois}(2)$

$$\frac{P(\text{Bonus} \cap \text{at least one arrived})}{P(\text{at least one arrived})} = \frac{P(X \geq 2 \cap X > 0)}{P(X > 0)}$$

$$= \frac{1 - \frac{2^0 e^{-2}}{0!} - \frac{2^1 e^{-2}}{1!}}{1 - \frac{2^0 e^{-2}}{0!}}$$

$$= \frac{1 - 3e^{-2}}{1 - e^{-2}}$$

$P(\text{bonus}) = P(2 \text{ or more})$

$= P(X \geq 2)$

$P(X \geq 2 \cap X > 0)$

$= P(X \geq 2)$

$= 1 - P(X \leq 1)$

$= 1 - P(X=0) - P(X=1)$

$P(X > 0) = 1 - P(X=0)$

TG

(c) (4 points) What is the probability that exactly 2 of the first 5 clerks received a bonus?

X = # of clerks who got bonus $\sim$ Binom$(5, p)$

p = probability of bonus = $\mathbb{P}(C \geq 2)$ where

C = # of customers in a shift $\sim$ Pois(2)

$p = \mathbb{P}(C \geq 2) = 1 - \mathbb{P}(C=0) - \mathbb{P}(C=1)$

$= 1 - 3e^{-2}$ (from prev part)

$$P(X=2) = \binom{5}{2}(1-3e^{-2})^2(3e^{-2})^3$$

(d) (4 points) What is the probability that the second bonus of the day is awarded to the sixth clerk?

$NB(2, 1-3e^{-2})$

Probability First bonus in first 5 clerks

Probability bonus on 6th clerk

$$P(\text{second bonus to } 6^{th} \text{ clerk}) = \binom{5}{1}(1-3e^{-2})(3e^{-2})^4 \cdot (1-3e^{-2})$$

TG

2. (a) (4 points) State the definition of a moment generating function (mgf) for a random variable.

If X is a RV

$$M(t) = E[e^{Xt}]$$

(b) (6 points) Let X be a random variable and let $M(t)$ be its mgf. Suppose that $M''(0) = 10$ and $E[X(1-X)] = -8$. Compute $\text{Var}(X)$.

$E[X^2] = 10$

$$E[X - X^2] = -8$$

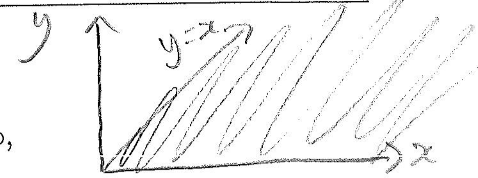
$$E[X] - E[X^2] = -8$$

$$E[X] = -8 + E[X^2]$$
$$= 2$$

$$\text{Var}(X) = E[X^2] - E[X]^2$$
$$= 10 - 2^2$$
$$= 6$$

T6

3. Let X and Y be random variables with joint pdf



$$f(x,y) = \begin{cases} \frac{e^{-x/2}}{2x} & \text{if } 0 \le y \le x < +\infty, \\ 0 & \text{otherwise.} \end{cases}$$

(a) (6 points) Find the marginal density of X. Be sure to specify the different cases if the marginal density is a piecewise function.

$$f_X(x) = \int_0^x f(x,y)\,dy$$

$$= \int_0^x \frac{e^{-x/2}}{2x}\,dy$$

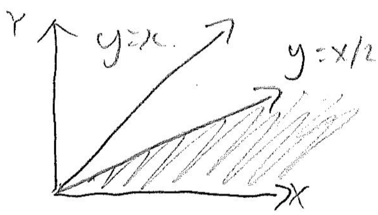
$$= \frac{e^{-x/2}}{2x} \cdot x - 0$$

$$= \frac{e^{-x/2}}{2}$$

for $x \ge 0$

$$f_X(x) = \begin{cases} \frac{e^{-x/2}}{2} & x \ge 0 \\ 0 & x < 0 \end{cases}$$

(b) (6 points) Compute $P(Y \le X/2)$.



Intuitively $\frac{1}{2}$ b/c we are bisecting the region.

$$P\left(Y \le \frac{X}{2}\right) = \lim_{a\to\infty} \int_0^a \int_0^{x/2} f(x,y)\,dy\,dx$$

$$= \lim_{a\to\infty} \int_0^a \frac{x}{2} \cdot \frac{e^{-x/2}}{2x}\,dx = \lim_{a\to\infty} \int_0^a \frac{e^{-x/2}}{4}\,dx$$

$$= \lim_{a\to\infty} \left(-\frac{e^{-a/2}}{2}\right) - \left(-\frac{e^{-0/2}}{2}\right) \quad (\to 0)$$

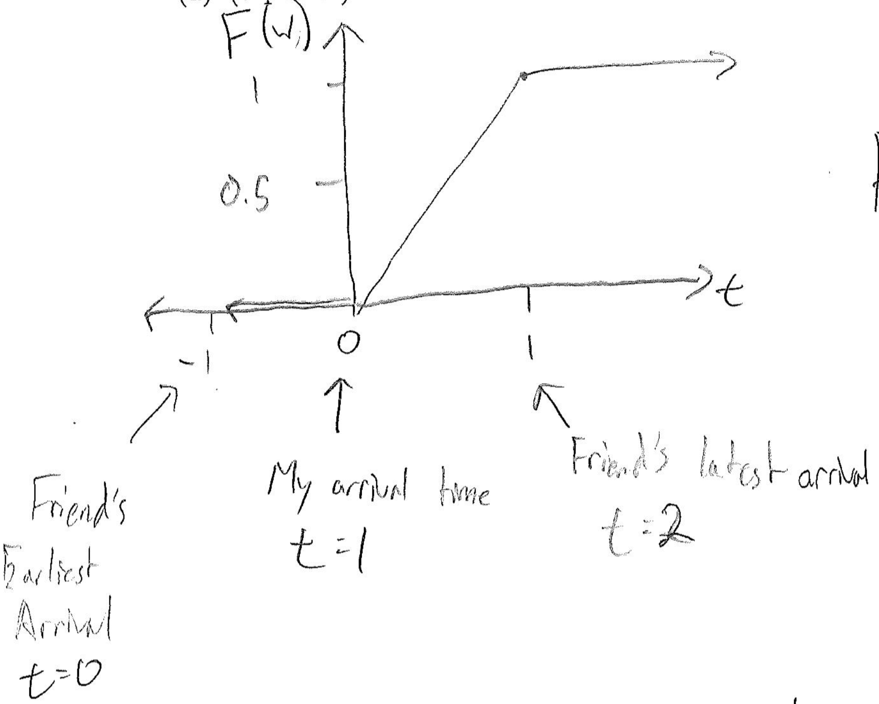
$$= \frac{1}{2}$$

TG

4. You and a friend are meeting at an agreed upon location. You will arrive at time $t = 1$, whereas your friend's arrival time is not so certain. We model your friend's arrival time with a continuous random variable which is uniform on $[0, 2]$.

Let W be the amount of time you wait for your friend once you arrive. Note that if your friend arrives before or at time $t = 1$, then $W = 0$.

(a) (7 points) Find the cdf of W.



$$F(W) = \begin{cases} 0 & W \le 0 \\ W & 0 < W \le 1 \\ 1 & 1 < W \end{cases}$$

(b) (5 points) Compute $E[W]$. Intuitively is $\frac{1}{4}$

t_f = time of friend's arrival

$$f(t_f) = \begin{cases} \frac{1}{2} & 0 \le t_f \le 2 \\ 0 & o/w \end{cases} \qquad f(w) = \begin{cases} \frac{1}{2} & W < 0 \\ \frac{1}{2} & 0 \le W \le 1 \\ 0 & o/W \end{cases}$$

$$E[W] = 0 \cdot P(W=0) + \int_0^1 w\, f(w)\, dw$$

$$= \int_0^1 \frac{1}{2} w\, dw = \frac{w^2}{4}\Big|_0^1 = \frac{1}{4} - 0 = \frac{1}{4}$$

(This page intensionally left blank.)