20W-MATH170E-1 Final

DEBORAH LIU

TOTAL POINTS

100 / 100

QUESTION 1

1 Question 1 10 / 10

✓ **- 0 pts** Correct

- 10 pts Does not use Bayes Rule or any conditional probability

QUESTION 2

2 Question 2 10 / 10

✓ **- 0 pts** Correct

- 1 pts Minor Computational Error on Part a)

- 1 pts Minor Computational Error on Part b)

- 2 pts Incorrect Formula for Variance

- 5 pts Severely incorrect probability computation in part b)

- 5 pts No answer to part b

QUESTION 3

3 Question 3 10 / 10

✓ **- 0 pts** Correct

- 5 pts Does not answer question asked in part a)

- 2 pts Incorrect computation for W2

- 2 pts Incorrect probability for X2

- 2 pts Incorrect Probability for W1

QUESTION 4

4 Question 4 10 / 10

✓ **- 0 pts** Correct

- 2 pts Incorrect PDF used

- 5 pts Severely Incorrect Part a)

- 5 pts Severely Incorrect Part b)

QUESTION 5

5 Question 5 10 / 10

✓ **- 0 pts** Correct

- 0 pts Minor Arithmetic error in a)

- 1 pts Error in Integral for a)

- 2 pts Major integration error in b)

- 3 pts Severely Incorrect part b)

- 0 pts Minor Arithmetic error in c)

- 4 pts Severely Incorrect part c)

QUESTION 6

6 Question 6 10 / 10

Part (a)

✓ **+ 1 pts** **Know the definition of the covariance (or its equivalent)**

✓ **+ 1 pts** **Know how to split the expectation using the independence**

✓ **+ 1 pts** **Correct answer**

Part (b)

✓ **+ 2 pts** **Know how to compute the probability using the independence**

✓ **+ 2 pts** **Correct answer**

+ 1 pts Minor mistakes

Part (c)

✓ **+ 1.5 pts** **Know how to compute $$E(Z)$$**

✓ **+ 1.5 pts** **Correct computation (including error carried forward)**

+ 1 pts Computation mistakes

QUESTION 7

7 Question 7 10 / 10

Part (a)

✓ **- 0 pts** **Everything correct**

- 1.5 pts Incorrect application of the law of total expectation

- 1.5 pts Incorrect application of the law of total variance

- 1 pts Minor mistakes

- 4 pts Showed some efforts but far from being complete

- **5 pts** None of above applies

Part (b)

✓ **- 0 pts Everything correct**

- **1 pts** Minor mistakes

- **1.5 pts** Did not correctly prove the distribution of $$Y$$

- **4 pts** The argument is far from being complete

- **5 pts** None of above applies

QUESTION 8

8 Question 8 **10 / 10**

Part (a)

✓ **- 0 pts Everything correct**

- **1 pts** Minor mistakes

- **2 pts** Major mistakes (such as confusing PDF by CDF)

Part (b)

✓ **- 0 pts Everything correct**

- **1 pts** Minor mistakes

- **1.5 pts** The answer is not simplified

- **3 pts** Major mistakes (such as providing a wrong answer without any intermediate computations, using PDF instead of CDF, etc)

- **5 pts** None of above

QUESTION 9

9 Question 9 **10 / 10**

✓ **- 0 pts Everything correct**

- **1 pts** Minor mistakes in computing $$M_Y(t)$$ (minor computation mistakes and/or the answer is not simplified)

- **2.5 pts** Major mistakes in computing $$M_Y(t)$$ (such as incomplete argument and/or answer)

- **2.5 pts** Incomplete argument for showing the distribution of $$Y$$

- **8 pts** Some efforts were made

QUESTION 10

10 Question 10 **10 / 10**

✓ **- 0 pts Everything correct**

- **1 pts** Minor mistakes

Take-Home Final

Name: Deborah Liu

Student ID: 2 0 5 1 4 0 7 2 5

Instructions:

- The take-home final will begin at 8 am March 20. You will be given **24 hours** to complete and submit your final. The submission window will be closed at 8 am March 21.
- **No late submission** will be considered. Make sure to allow enough time to complete and submit your works. **You must take the final exam in order to pass the class.** Make-ups for the final exam are permitted only under exceptional circumstances, as outlined in the UCLA student handbook.
- The exam will be open book/open notes. More specifically, you may **only use the textbook, lecture notes, and/or any materials uploaded to the CCLE course web page.** You must **show your work** to receive credit.
- You must **sign the code of conduct:**

> I assert, on my honor, that I have not received assistance of any kind from any other person while working on the final and that I have not used any non-permitted materials or technologies during the period of this evaluation.
>
> Signature: Deborah Liu

Any deviation from the rules may render your exam void. Also, if needed, you may be contacted after the exam and asked for additional explanations of solutions for problems on the final.

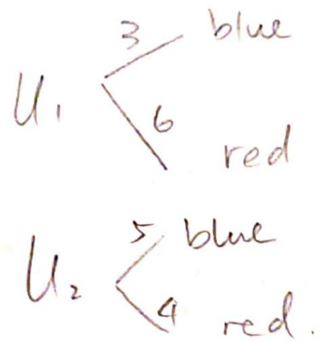
- **A Gradescope link for submitting your work will be provided.** Your submission should meet a set of criteria:
  (a) Your submission must be a single PDF file.
  (b) The code of conduct, your name, UID, and either physical or electronic signature must appear on the first page. (See above for an example.)
  (c) Starting from page 2, your solution to each of the problems must appear on a single page, in the order of the numbering. (For example, your solution to Question 1 must appear on page 2, Question 2 on page 3, and so forth.)

  There will be several ways to achieve this. The following is a set of common examples:
  - The exam template file will be designed to satisfy the above criteria. So you may simply print it out, fill in the necessary forms, and write down your solution to each of the problems. And then you may either scan it or take a (high-resolution and high-contrast) picture of it.
  - You may use letter size blank papers as soon as all the above criteria are met.
  - You may directly write on the exam PDF file, such as using a tablet.
  - You may use a word processor or LaTeX to prepare your submission electronically.

1. (10 pts) Suppose that:

- Urn U_1 contains three blue balls and six red balls, and
- Urn U_2 contains five blue balls and four red balls.

Suppose we draw one ball at random from each urn. If the two balls drawn have different colors, what is the probability that the blue ball came from U_1?



P(two balls have different colors)

$= P(\text{blue from } U_1, \text{ red from } U_2) + P(\text{red from } U_1, \text{blue from } U_2)$

$= \frac{1}{3} \cdot \frac{4}{9} + \frac{2}{3} \cdot \frac{5}{9} = \frac{4+10}{27} = \frac{14}{27}$.

$P(\text{blue from } U_1, \text{ red from } U_2) = \frac{1}{3} \cdot \frac{4}{9} = \frac{4}{27}$

$P(\text{blue from } U_1 \mid \text{two balls have different colors})$

$= \dfrac{\frac{4}{27}}{\frac{14}{27}} = \frac{4}{14} = \frac{2}{7}$

2. (10 pts) Suppose that the random variable X takes only integer values and has the MGF of the form

$$M_X(t) = \frac{4}{2-e^t} - \frac{6}{3-e^t}.$$

(a) Compute the variance of X.

(b) Find $P(X \le 2)$.

a) $$E(X) = \frac{d}{dt} M_X(t)\Big|_{t=0} = \left[-\frac{4}{(2-e^t)^2} \cdot (-e^t) + \frac{6}{(3-e^t)^2}(-e^t) \right]\Bigg|_{t=0}$$

$$= \left[e^t \left(\frac{4}{(2-e^t)^2} - \frac{6}{(3-e^t)^2} \right) \right]_{t=0}$$

$$= 1\left(\frac{4}{(2-1)^2} - \frac{6}{(3-1)^2} \right)$$

$$= 4 - \frac{6}{4} = 4 - \frac{3}{2} = \frac{5}{2}$$

$$E(X^2) = \frac{d^2}{dt^2} M_X(t)\Big|_{t=0}$$

$$= \left(e^t \left(\frac{4}{(2-e^t)^2} - \frac{6}{(3-e^t)^2} \right) + e^t \left(\frac{4}{(2-e^t)^3}(-2)(-e^t) - \frac{6}{(3-e^t)^3}(-2)(-e^t) \right) \right)\Bigg|_{t}$$

$$= \frac{5}{2} + 1 \cdot \left(\frac{4}{(2-1)^3} \cdot 2 - \frac{6}{(3-1)^3} \cdot 2 \right)$$

$$= \frac{5}{2} + \frac{8}{1} - \frac{12}{8} = \frac{5}{2} + 8 - \frac{3}{2} = 9$$

$$Var(X) = E(X^2) - (E(X))^2 = 9 - \frac{25}{4} = \frac{11}{4}$$

b) $$M_X(t) = \frac{2}{1-\frac{e^t}{2}} - \frac{2}{1-\frac{e^t}{3}} \qquad \frac{a}{1-r} = \sum_{n=0}^{\infty} ar^n$$

$$= 2\sum_{n=0}^{\infty} \left(\frac{e^t}{2}\right)^n - 2\sum_{n=0}^{\infty} \left(\frac{e^t}{3}\right)^n = 2\left(\sum_{n=0}^{\infty} \left(\frac{e^t}{2}\right)^n - \sum_{n=0}^{\infty} \left(\frac{e^t}{3}\right)^n \right)$$

$$M_X(t) = E(e^{tX}) = \sum e^{tx} \cdot p(x) = 2\left[\left(1 + \frac{e^t}{2} + \frac{e^{2t}}{4} + \cdots\right) - \left(1 + \frac{e^t}{3} + \frac{e^{2t}}{9} + \cdots\right) \right]$$

for $X=2$ the term is $2\left(\frac{e^{2t}}{4} - \frac{e^{2t}}{9}\right) = e^{2t}\left(\frac{1}{2} - \frac{2}{9}\right)$ $P(X=2) = \frac{5}{18}$

$P(X=1) = 2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{2}{6}$ $P(X \le 2) = \frac{5}{18} + \frac{2}{6} = \frac{11}{18}$

3. (10 pts) A fisherman goes to the pond at 8 a.m. and catches fish according to a Poisson process with a rate of $\lambda = 0.6$ catches per hour. Let X_1 be the number of catches from 8 a.m. to 9 a.m., and let X_2 be the number of catches from 9 a.m. to 11 a.m.

(a) Find $P(X_1 + X_2 = 5,\ X_2 = k)$ for $k = 0, \ldots, 5$.

(b) Let W_i be the time, in hours, it takes the fisherman to catch i fish. Find $P(W_1 \le 1, W_2 > 3)$.

HINT: Rephrase the event $\{W_1 \le 1, W_2 > 3\}$ in terms of X_1 and X_2.

a) $X_1 \sim \text{Poisson}(0.6) \qquad X_2 \sim \text{Poisson}(1.2)$

$$P(X_1+X_2=5, X_2=k) = P(X_1=5-k, X_2=k) = P(X_1=5-k)P(X_2=k)$$

$$\boxed{= e^{-0.6}\cdot\frac{(0.6)^{5-k}}{(5-k)!}\cdot e^{-1.2}\frac{(1.2)^k}{k!}}$$

$$\sum_{k=0}^{5} P(X_1=5-k, X_2=k) = e^{-0.6}\cdot e^{-1.2}\left(\frac{0.6^5}{5!} + \frac{0.6^4}{4!}1.2 + \frac{0.6^3}{3!}\frac{(1.2)^2}{2!} + \frac{0.6^2}{2!}\frac{(1.2)^3}{3!} + 0.6\frac{(1.2)^4}{4!} + \frac{(1.2)^5}{5!}\right)$$

$$= e^{-1.8}\frac{(0.6+1.2)^5}{5!} = e^{-1.8}\frac{(1.8)^5}{5!}$$

b) $W_1 \le 1$: at least 1 fish before 9 a.m

$W_2 > 3$: at most 1 fish before 11 a.m.

$\therefore$ exactly 1 fish between 8-9 a.m.

exactly 0 fish between 9-11 a.m.

$$P(X_1=1, X_2=0) = e^{-0.6}\frac{(0.6)^1}{1!}\cdot e^{-1.2}\cdot\frac{(1.2)^0}{0!}$$

$$= e^{-0.6}\cdot 0.6\cdot e^{-1.2}$$

$$= 0.6\,e^{-1.8}$$

4. (10 pts) The amount X, in GBs, of mobile data Ahmed uses in a given month is uniformly distributed between 0 and 10. The mobile data plan Ahmed is subscribed to charges Y dollars per month, where

$$Y = \begin{cases} 20, & X < 3, \\ 10\lfloor X \rfloor, & 3 \le X < 6, \\ 10X, & X \ge 6. \end{cases}$$

Here, $\lfloor x \rfloor$ denotes the floor of x.

(a) Find $P(Y < 45)$.

(b) Compute the expected value of Y.

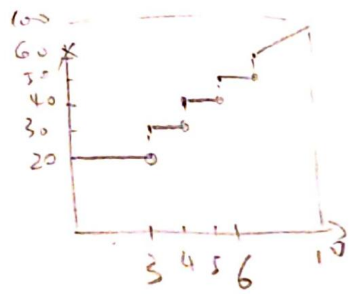
(a)

$$P(Y<45) = P(Y<45 \mid X<3)P(X<3) + P(Y<45 \mid 3\le X<6)P(3\le X<6) + P(Y<45 \mid X\ge 6)P(X\ge 6).$$

$$= \frac{3}{10} + P(3 \le X < 5 \mid 3 \le X < 6)\cdot\frac{3}{10}$$

$$= \frac{3}{10} + \frac{2}{3}\cdot\frac{3}{10} = \frac{3}{10} + \frac{2}{10} = \frac{5}{10} = \frac{1}{2}$$

b)



$$E(Y) = 20\cdot P(Y=20) + 30\cdot P(30) + 40\cdot P(40) + 50\cdot P(50) + \int_{60}^{100} y f_Y(y)\,dy.$$

for $X \ge 6$, $X = \frac{Y}{10}$

$$f_Y(y) = v'(y)\cdot f_X(v(y)) = \frac{1}{10}\cdot\frac{1}{10} = \frac{1}{100}$$

$$E(Y) = 20\cdot\frac{3}{10} + 30\cdot\frac{1}{10} + 40\cdot\frac{1}{10} + 50\cdot\frac{1}{10} + \int_{60}^{100}\frac{y}{100}\,dy$$

$$= 6 + 3 + 4 + 5 + \left[\frac{y^2}{200}\right]_{60}^{100} = 6+3+4+5+32 = 18+32 = 50$$

5. (10 pts) Let X and Y have the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} cx, & 0 < x < 1,\ 0 < y < 1 - x^2, \\ 0, & \text{elsewhere,} \end{cases}$$

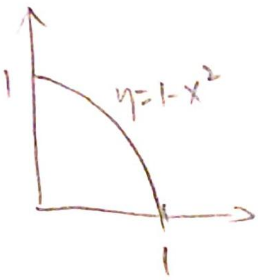
where c is a constant.

(a) Determine the value of c.

(b) Find the marginal PDF of X.

(c) Compute $E(X^2Y)$.

(a)



$$\int_0^1\int_0^{1-x^2} cx\,dy\,dx = c\int_0^1 x(1-x^2)\,dx$$

$$= c\left(\int_0^1 x\,dx - \int_0^1 x^3\,dx\right)$$

$$= c\left(\left[\frac{x^2}{2}\right]_0^1 - \left[\frac{x^4}{4}\right]_0^1\right)$$

$$= c\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{c}{4} = 1 \quad c=4$$

b) $f_X(x) = \int_0^{1-x^2} cx\,dy = 4(x)(1-x^2) = 4(x-x^3)$

c) $$E(X^2Y) = \int_0^1\int_0^{1-x^2} x^2y \cdot 4x\,dy\,dx = \int_0^1 4x^3\left[\frac{y^2}{2}\right]_0^{1-x^2} dx$$

$$= \int_0^1 \frac{4x^3(1-x^2)^2}{2}\,dx$$

$$= \int_0^1 \frac{4x^3(1-2x^2+x^4)}{2}\,dx$$

$$= \int_0^1 2x^3 - 4x^5 + 2x^7\,dx$$

$$= \left[\frac{x^4}{2} - \frac{4x^6}{6} + \frac{x^8}{4}\right]_0^1$$

$$= \frac{1}{2} - \frac{4}{6} + \frac{1}{4} = \frac{6-8+3}{12} = \frac{1}{12}.$$

6. (10 pts) Let X and Y be independent exponential random variables with mean $\theta = 4$.

(a) Compute the covariance between XY and X.

(b) Let $Z = \max\{X, Y\}$. Find the PDF of Z.

(c) Let Z be as in the previous part. Compute the expected value of Z.

a) $\mathrm{Cov}(XY, X) = E(XY \cdot X) - E(XY)E(X)$

$= E(X^2)E(Y) - (E(X))^2 E(Y)$

$= E(Y)(E(X^2) - (E(X))^2)$

$= E(Y)\mathrm{Var}(X) = 4 \cdot 4^2 = 64$

b) $F_Z(z) = P(Z \le z) = P(\max\{X, Y\} \le z)$

$= P(X \le z)P(Y \le z)$

$= F_X(z)F_Y(z)$

$= (1 - e^{-\frac{z}{4}})(1 - e^{-\frac{z}{4}})$

$= (1 - e^{-\frac{z}{4}})^2$

$f_Z(z) = \frac{d}{dz}F_Z(z) = 2(1 - e^{-\frac{z}{4}}) \cdot (\frac{1}{4}e^{-\frac{z}{4}})$

$= \frac{1}{2}e^{-\frac{z}{4}}(1 - e^{-\frac{z}{4}}) = \frac{1}{2}e^{-\frac{z}{4}} - \frac{1}{2}e^{-\frac{z}{2}}$

c) $E(Z) = \int_0^\infty \frac{1}{2}e^{-\frac{z}{4}}(1 - e^{-\frac{z}{4}}) \cdot z\,dz = \int_0^\infty \frac{1}{2}e^{-\frac{z}{4}} \cdot z\,dz - \int_0^\infty \frac{1}{2}e^{-\frac{z}{2}} z\,dz$

$= 8 - 2 = 6.$

7. (10 pts) Suppose that X have a gamma distribution with parameters $\alpha = 2$ and $\theta = 3$, and suppose that the conditional distribution of Y, given $X = x$, is uniform between 0 and x.

(a) Find $E(Y)$ and $\mathrm{Var}(Y)$.

(b) Find the MGF of Y. What is the distribution of Y?

a) $E(Y) = E(E(Y|X)) = E\left(\frac{X}{2}\right) = \frac{1}{2}E(X) = \frac{1}{2}\cdot 2\cdot 3 = 3$

$$Var(Y) = E(Var(Y|X)) + Var(E(Y|X))$$
$$= E\left(\frac{X^2}{12}\right) + Var\left(\frac{X}{2}\right)$$
$$= \frac{1}{12}E(X^2) + \frac{1}{4}Var(X)$$
$$= \frac{1}{12}\left(Var(X) + (E(X))^2\right) + \frac{1}{4}\alpha\theta^2$$
$$= \frac{1}{12}\left(\alpha\theta^2 + (\alpha\theta)^2\right) + \frac{1}{4}\alpha\theta^2$$
$$= \frac{1}{12}(18 + 36) + \frac{1}{4}\cdot 2\cdot 9$$
$$= \frac{54}{12} + \frac{9}{2} = \frac{9}{2} + \frac{9}{2} = 9$$

b) $E(e^{tY}) = E(E(e^{tY}|X)) = E\left(\frac{e^{tX} - e^{t0}}{t(X)}\right) = E\left(\frac{e^{tX}-1}{Xt}\right)$

$$= \int_0^\infty \frac{e^{tX}-1}{Xt}\cdot\frac{X^{\alpha-1}e^{-\frac{X}{\theta}}}{\Gamma(\alpha)\theta^\alpha}\,dx = \frac{1}{9\Gamma(\alpha)}\int_0^\infty \frac{e^{tX}-1}{t\cdot X}\, X e^{-\frac{X}{3}}\,dx$$
$$= \frac{1}{9}\int_0^\infty \frac{1}{t}\left(e^{X(t-\frac{1}{3})} - e^{-\frac{X}{3}}\right)dx$$
$$= \frac{1}{9t}\left[e^{X(t-\frac{1}{3})}\cdot\frac{1}{t-\frac{1}{3}} + 3e^{-\frac{X}{3}}\right]_0^\infty$$
$$= \frac{1}{9t}\left(\frac{1}{\frac{1}{3}-t} - 3\right) = \frac{1}{9t}\left(\frac{3-3(1-3t)}{1-3t}\right) = \frac{1}{9t}\,\frac{9t}{1-3t}$$
$$= \frac{1}{(1-3t)}$$

Y is exponential distribution with mean $= 3$

8. (10 pts) Let X be a continuous random variable having the CDF

$$F_X(x) = 1 - e^{-e^x}.$$

(a) Find the PDF of $Y = e^X$.

(b) Let Z have a uniform distribution over $(0,1)$. Find a function $G(z)$ such that $G(Z)$ has the same distribution as X.

a) $f_X(x) = -e^{-e^x} \cdot (-e^x)$

$= e^{-e^x} e^x = e^{x-e^x}$

$Y = e^x \quad X = \ln Y = v(y) \quad v'(y) = \frac{1}{y}$

$\therefore f_Y(y) = \frac{1}{y} \cdot e^{\ln y - e^{\ln y}} = \frac{1}{y} \cdot e^{\ln y - y} = \frac{1}{y}(y e^{-y})$

$= e^{-y}$

b) $f_Z(z) = 1 \qquad G(z) = X \qquad Z = G^{-1}(X)$

$f_X(x) = (G^{-1}(x))' \cdot f_Z(G^{-1}(x))$

$= (G^{-1}(x))' = e^{x-e^{-x}}$

$z = G^{-1}(x) = \int e^{x-e^{x}} dx = 1 - e^{-e^x}$

$1 - z = e^{-e^x} \qquad \ln(1-z) = -e^x$

$-\ln(1-z) = e^x$

$\boxed{\ln(-\ln(1-z))} = X = G(z)$

Another way to solve it is $F_Z(z) = z \qquad F_Z(G^{-1}(x)) = G^{-1}(x) = F_X(x)$

$z = G^{-1}(x) = 1 - e^{-e^x}$

$1 - z = e^{-e^x} \qquad \ln(1-z) = -e^x \quad \ln(-\ln(1-z)) = X = G(z)$

9. (10 pts) Let $a \geq 2$ be an integer, and let X be uniformly distributed over the set of integers $\{0, 1, \ldots, a-1\}$. Then it is easy to check that the MGF of X is given by

$$M_X(t) = \frac{1}{a}(1 + e^t + \cdots + e^{(a-1)t}) = \begin{cases} \frac{1}{a} \cdot \frac{e^{at}-1}{e^t - 1}, & t \neq 0, \\ 1, & t = 0. \end{cases} \qquad (\star)$$

Now let $n \geq 1$ be an integer, and let $X_0, \ldots, X_{n-1}$ be independent random variables identically distributed as X. Set

$$Y = \sum_{i=0}^{n-1} X_i a^i.$$

(a) Use $(\star)$ to compute the MGF of Y.

(b) Show that Y is uniformly distributed over the set of integers $\{0, 1, \ldots, a^n - 1\}$.

when $t \neq 0$.

a) $E(e^{tY}) = E\left(e^{(\sum_{i=0}^{n-1} X_i a^i)t}\right) = E\left(e^{X_0 t} \cdot e^{aX_1 t} \cdot e^{a^2 X_2 t} \cdots\right)$

$= E(e^{X_0 t}) E(e^{aX_1 t}) \cdots E(e^{a^{n-1}X_{n-1} t})$

$= \frac{1}{a} \frac{\cancel{e^{at}-1}}{e^t - 1} \cdot \frac{1}{a} \frac{e^{a^2 t} \cancel{-1}}{\cancel{e^{at}-1}} \cdot \cdots \cdot \frac{1}{a} \cdot \frac{e^{a^n t} - 1}{\cancel{e^{a^{n-1} t}-1}}$

$= \frac{1}{a^n} \frac{e^{a^n t} - 1}{e^t - 1} = M_Y(t)$

b) let Z be uniformly distributed over the set $\{0, 1, \ldots$
for any $z \in \{0, \ldots, a^n - 1\}$, $P(Z = z) = \frac{1}{a^n}$

$M_Z(t) = \frac{1}{a^n} (1 + e^t + \cdots e^{(a^n - 1)t})$

$= \frac{1}{a^n} \cdot \frac{e^{a^n t} - 1}{e^t - 1} = M_Y(t)$

10. (10 pts) A postal sorting equipment processes mails, one at a time. The processing times, in milliseconds, of different mails are independent and uniformly distributed in $[10, 70]$. Use the Central Limit Theorem to approximate the probability that the number of mails processed within 3300 milliseconds is at least 75.

HINT: Rewrite the event in terms of the time Y, in milliseconds, until the first 75 mails are processed.

$X_i \sim \mathcal{U}((10,70)$ $\quad \mu = \frac{10+70}{2} = 40 \quad \sigma^2 = \frac{1}{12}(70-10)^2 = 300$

$Y = \sum_{i=1}^{75} X_i \quad W = \frac{Y - n\mu}{\sqrt{n}\,\sigma} = \frac{Y - 75 \cdot 40}{\sqrt{75 \cdot 300}} = \frac{Y-3000}{150}$

$P(Y \leq 3300)$

$= P\left(W \leq \frac{3300-3000}{150}\right)$

$= P(W \leq 2) = \Phi(2)$