Midterm — February 7th

Please provide complete and well-written solutions to the following exercises. All answers must be properly explained and justified.

Exercise 1. (/2) You have a box with 6 red balls, 6 blue balls and 6 yellow balls. Roll a fair, six-sided die. Let i be the resulting number. Then draw i balls uniformly at random from the box without replacement.

(i) What is the probability you draw at least two balls of the same color, given you roll a 4?

(ii) What is the probability you draw at least two balls of the same color?

Exercise 2. (/1) There are n different cars labeled with numbers 1 to n. These cars are parked in n consecutive parking spaces from left to right in a uniformly random order. What is the probability that the cars labeled 1 and 2 are parked next to each other (meaning that car 2 is directly to the left or to the right of car 1)?

Exercise 3. (/2) Let D_1, D_2, D_3 be three 4-sided dice whose sides have been labeled as follows:
$$D_1 : 0, 3, 3, 3, \qquad D_2 : 2, 2, 2, 5, \qquad D_3 : 1, 1, 4, 6.$$

The three dice are rolled at random. Let A be the event that the outcome of D_1 is larger than D_2, let B the event that the outcome of D_2 is larger than D_3, and let C the event that the outcome of D_3 is larger than the outcome of D_1. Compute $\mathbb{P}[A], \mathbb{P}[B], \mathbb{P}[C]$. What is the best die?

Exercise 4. (/1) In your pocket is a random number N of coins, where N has the Poisson distribution with parameter λ. You toss each coin once, with heads showing with probability p each time. Show that the total number of heads has the Poisson distribution with parameter λp.

Exercise 5. (/2) In a certain game the first contestant chooses either number 2 or number 3 with equal probability. If the number chosen is $k \in \{2, 3\}$, the second contestant will choose an integer number between 1 and k uniformly at random. Let X be the number chosen by the first contestant and Y be the number chosen by the second contestant.

(i) Find the joint probability mass function of the pair (X, Y).

(ii) Compute $\mathbb{E}[Y]$.

Exercise 6. (/2) You are lost in a national park and you are trying to find a ranger to ask for directions. There are two types of rangers: Two-thirds of them will give you a correct answer with probability $\frac{3}{4}$, independently each time you ask. The remaining third of the rangers will always give you the wrong answer.

(i) You find a ranger, without knowing to which type they belong. You ask whether the exit from the park is East or West, and their answer is East. What is the probability that it is correct?

(ii) You ask the same ranger again and receive the same answer. Show that the probability that it is correct is $\frac{1}{2}$.
Hint: Let T be the event that the answer is correct and let E_2 be the event that you get twice the same answer. Use Bayes' rule to compute $\mathbb{P}\left[T|E_2\right]$, assuming that East and West are equally likely to be actually the correct answers.

(iii) You ask the same person again for the third time and receive the same answer. Now what is the probability that it is correct?