Final exam — March 14th

Please provide complete and well-written solutions to the following exercises. All answers must be properly explained and justified.

By taking this exam, you implicitly agree to solve the exercises and write your answers on your own without using any means of cheating.

Exercise 1. (/1) Let X and Y be independent discrete random variables, X having the geometric distribution with parameter p and Y having the geometric distribution with parameter r. What is the distribution of $\min\{X, Y\}$?

Exercise 2. (/3) You arrive at the opening of a new cafe, which has only two chairs. Unfortunately, there are three people ahead of you in the line. You know that the number of minutes that a guest spends in a cafe is a geometric random variable with mean 2, and that the time each guest spends in the cafe is independent of all other guests. What is the probability that you will have to wait strictly more than n minutes.

Exercise 3. (/2) The number of bikes that arrive for repair in a certain bike shop is a Poisson random variable with mean 1. For $k \geq 1$, given that exactly k bikes arrive, the total time T that it takes to repair all these bikes is a uniform random variable on the interval $[0, k+1]$. Compute the probability $\mathbb{P}[T \leq 1]$ and the expectation $\mathbb{E}[T]$.

Exercise 4. (/3) Let T be a triangle with vertices $(-1, 0)$, $(1, 0)$, and $(0, 1)$. Let the random variables X and Y be jointly continuous with joint pdf

$$f_{X,Y}(x, y) := \begin{cases} C|xy| & : \quad (x, y) \in T, \\ 0 & : \quad (x, y) \notin T. \end{cases}$$

(i) Find the value of the constant C.

(ii) Are X and Y independent?

(iii) Find the marginal pdf of X and the conditional pdf of X given $Y = y$.

Exercise 5. (/2) Suppose that a basketball player can make a free throw 60% of the time. Let X equal the minimum number of free throws that this player must attempt to make a total of 30 shots.

(a) Give the mean, variance, and standard deviation of X.

(b) Find an exact expression for $\mathbb{P}[70 < X \leq 100]$. Use normal approximation to estimate this probability.

Exercise 6. (/2) Flaws in a certain type of drapery material appear on the average of one in 150 square feet. Assuming independence, evaluate the probability of at most one flaw appearing in 225 square feet.

Exercise 7. (/4) Given a random variable X with expectation μ and variance σ^2, define its skewness and its kurtosis, respectively, as

$$\mathrm{skw}(X) := \sigma^{-3}\mathbb{E}\left[(X-\mu)^3\right] \qquad \text{and} \qquad \mathrm{kur}(X) := \sigma^{-4}\mathbb{E}\left[(X-\mu)^4\right].$$

(i) Compute the skewness and the kurtosis of a normal random variable.

(ii) Compute the skewness and the kurtosis of an exponential random variable.

(iii) Given iid random variables $X_1, \ldots, X_n$, compute the skewness and the kurtosis of their sum $S_n := X_1 + \ldots + X_n$. To what do they converge as $n \uparrow \infty$? Does it surprise you?

Exercise 8. (/3) Given $a, b > 0$, let X, Y be random variables with $\Gamma(a, 1)$ and $\Gamma(b, 1)$ distributions, respectively.

(i) Show that $X + Y$ and $\frac{X}{X+Y}$ are independent and find their distribution.

(ii) Deduce that $\mathbb{E}\left[\frac{X}{X+Y}\right] = \frac{\mathbb{E}[X]}{\mathbb{E}[X]+\mathbb{E}[Y]}$.