20F-MATH170E-1 Midterm 1

TOTAL POINTS

100 / 100

QUESTION 1

1 1-a **5 / 5**

✓ **- 0 pts** Correct

- 3 pts independence reasoning is not correct

- 5 pts wrong solution with fundamental mistake

QUESTION 2

2 1-b **5 / 5**

✓ **- 0 pts** Correct

- 1 pts minor mistake

- 3 pts wrong conditional probability formula

QUESTION 3

3 1-c **5 / 5**

✓ **- 0 pts** Correct

- 1 pts Why P(B')=0.6. Need to use complement rule.

- 4 pts A' and B' are not independent

- 3 pts wrong conditional probability formula

- 2 pts Faulty De morgan

- 1 pts why P(A' intersection B') = 0.5?

QUESTION 4

4 2-a **5 / 5**

✓ **+ 5 pts** Correct

+ 2 pts Correct total number 2^6

- 1 pts Minor mistake

QUESTION 5

5 2-b **5 / 5**

✓ **+ 5 pts** Correct

+ 2 pts Correct total numbers 2^6

- 1 pts Minor mistake

QUESTION 6

6 3 **10 / 10**

✓ **- 0 pts** Correct

- 4 pts order of A and B not counted

- 7 pts no explanation

QUESTION 7

7 4 **15 / 15**

✓ **+ 15 pts** Correct

+ 10 pts Applied Bayes theorem

- 3 pts Minor mistake

QUESTION 8

8 5 **20 / 20**

✓ **+ 20 pts** Correct

+ 10 pts binomial is correct.

+ 3 pts partition of events are correct

+ 3 pts conditional probabilities are correct

+ 4 pts final answer is correct

+ 6 pts Answer is correct but conditional probabilities not explained properly

+ 5 pts partial credit for binomial

+ 7 pts binomial missing/wrong coefficient

+ 0 pts wrong

+ 2 pts incorrect binomial idea

+ 3 pts regrade

QUESTION 9

9 6-a **10 / 10**

✓ **+ 10 pts** Correct

+ 5 pts Partially correct

- 1 pts Minor mistake

QUESTION 10

10 6-b **5 / 5**

✓ **+ 5 pts** Correct

+ 3 pts Partially correct

- 1 pts Minor mistake

QUESTION 11

11 6-c **5 / 5**

✓ **+ 5 pts** Correct

+ 3 pts Partially correct

- 1 pts Minor mistake

QUESTION 12

12 6-d **5 / 5**

✓ **+ 5 pts** Correct

+ 3 pts Partially correct

- 1 pts Minor mistake

QUESTION 13

13 6-e **5 / 5**

✓ **+ 5 pts** Correct

+ 3 pts Partially correct

- 1 pts Minor mistake

QUESTION 14

14 honesty statement **0 / 0**

✓ **+ 0 pts** Correct

1. (15 points) Suppose that events A and B satisfy

$$P(A) = 0.3, \quad P(B) = 0.4, \quad P(A \cup B) = 0.5.$$

(a) (5 points) Are A and B independent?

Events A and B are independant if $P(A \cap B) = P(A) \cdot P(B)$

$= 0.3 \cdot 0.4$

$P(A \cap B) = 0.12 \neq 0.2$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cap B) = P(A) + P(B) - P(A \cup B)$

$0.3 + 0.4 - 0.5$

$P(A \cap B) = 0.2$

Thus, A and B are not independent

(b) (5 points) Compute $P(A|B)$.

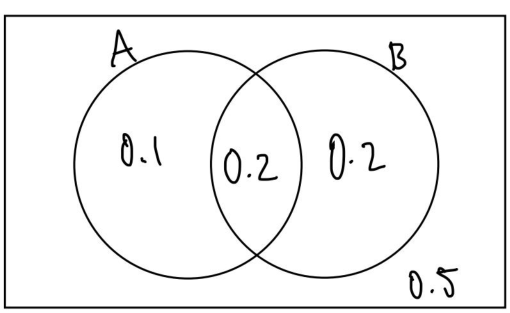
$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.2}{0.4}$$

$P(A|B) = 0.5$

(c) (5 points) Compute $P(A'|B')$. Here, A' and B' denote the complement of A and B, respectively.

Complement rule

$$P(A'|B') = \frac{P(A' \cap B')}{P(B')} = \frac{1 - P(A \cup B)}{1 - P(B)}$$



$$= \frac{1 - 0.5}{1 - 0.4} = \frac{0.5}{0.6}$$

$P(A'|B') = \frac{5}{6}$

1 1-a **5 / 5**

✓ **- 0 pts** Correct

- 3 pts independence reasoning is not correct

- 5 pts wrong solution with fundamental mistake

1. (15 points) Suppose that events A and B satisfy

$$P(A) = 0.3, \quad P(B) = 0.4, \quad P(A \cup B) = 0.5.$$

(a) (5 points) Are A and B independent?

Events A and B are independant if $P(A \cap B) = P(A) \cdot P(B)$

$= 0.3 \cdot 0.4$

$P(A \cap B) = 0.12 \neq 0.2$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cap B) = P(A) + P(B) - P(A \cup B)$

$0.3 + 0.4 - 0.5$

$P(A \cap B) = 0.2$

Thus, A and B are not independent

(b) (5 points) Compute $P(A|B)$.

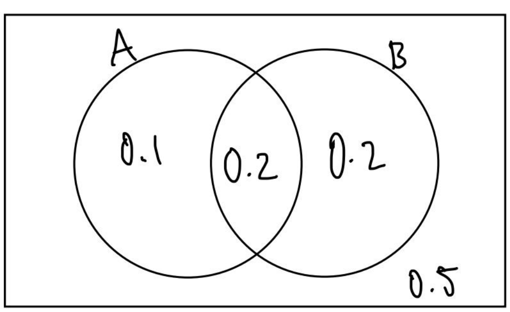
$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.2}{0.4}$$

$P(A|B) = 0.5$

(c) (5 points) Compute $P(A'|B')$. Here, A' and B' denote the complement of A and B, respectively.

complement rule

$$P(A'|B') = \frac{P(A' \cap B')}{P(B')} = \frac{1 - P(A \cup B)}{1 - P(B)}$$



$$= \frac{1 - 0.5}{1 - 0.4} = \frac{0.5}{0.6}$$

$P(A'|B') = \frac{5}{6}$

2 1-b 5 / 5

✓ **- 0 pts** Correct

- 1 pts minor mistake

- 3 pts wrong conditional probability formula

1. (15 points) Suppose that events A and B satisfy

$$P(A) = 0.3, \quad P(B) = 0.4, \quad P(A \cup B) = 0.5.$$

(a) (5 points) Are A and B independent?

Events A and B are independendent if $P(A \cap B) = P(A) \cdot P(B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 0.3 \cdot 0.4$

$P(A \cap B) = P(A) + P(B) - P(A \cup B)$

$P(A \cap B) = 0.12 \neq 0.2$

$0.3 + 0.4 - 0.5$

$P(A \cap B) = 0.2$

Thus, A and B are not independent

(b) (5 points) Compute $P(A|B)$.

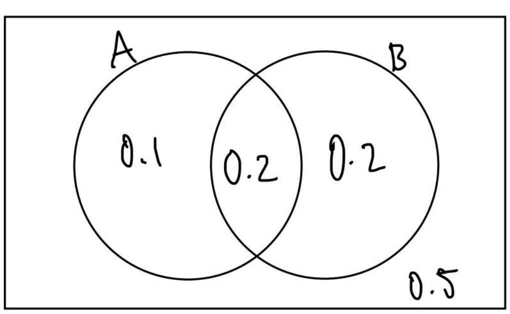
$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.2}{0.4}$$

$P(A|B) = 0.5$

(c) (5 points) Compute $P(A'|B')$. Here, A' and B' denote the complement of A and B, respectively.

complement rule

$$P(A'|B') = \frac{P(A' \cap B')}{P(B')} = \frac{1 - P(A \cup B)}{1 - P(B)}$$



$$= \frac{1 - 0.5}{1 - 0.4} = \frac{0.5}{0.6}$$

$P(A'|B') = \frac{5}{6}$

3 1-c **5 / 5**

- ✓ **- 0 pts** Correct
- **- 1 pts** Why P(B')=0.6. Need to use complement rule.
- **- 4 pts** A' and B' are not independent
- **- 3 pts** wrong conditional probability formula
- **- 2 pts** Faulty De morgan
- **- 1 pts** why P(A' intersection B') = 0.5?

2. (10 points) A fair coin (i.e. comes up head with probability 1/2) is thrown for six times.
(a) (5 points) Compute the probability that exactly four many heads appear.

total number of ways the coin could be thrown $= 2^6$

number of ways we can choose 4 heads out of 6 spots

$$P(4 \text{ heads}) = \frac{{}_6C_4}{2^6} = \frac{\frac{6!}{4!(6-4)!}}{64} = \frac{\frac{6\times 5}{2}}{64} = \frac{15}{64}$$

$$\boxed{P(4 \text{ heads}) = \frac{15}{64}}$$

(b) (5 points) Compute the probability that head appears both at the second and third trials.

let us fix the two spots $\underline{2}\ \underline{H}\ \underline{H}\ \underline{2}\ \underline{2}\ \underline{2}$

and vary the rest of the 4 spots

$$P(\text{Head at second } \& \text{ third}) = \frac{2^4}{2^6} = \frac{1}{4}$$

$$\boxed{P(\text{head at } 2^{nd} \ \& \text{ third}) = \frac{1}{4}}$$

4 2-a **5 / 5**

- ✓ **+ 5 pts** Correct
- **+ 2 pts** Correct total number 2^6
- **- 1 pts** Minor mistake

2. (10 points) A fair coin (i.e. comes up head with probability 1/2) is thrown for six times.
(a) (5 points) Compute the probability that exactly four many heads appear.

total number of ways the coin could be thrown $= 2^6$

number of ways we can choose 4 heads out of 6 spots

$$P(4 \text{ heads}) = \frac{6C4}{2^6} = \frac{\frac{6!}{4!(6-4)!}}{64} = \frac{\frac{6\times 5}{2}}{64} = \frac{15}{64}$$

$$\boxed{P(4 \text{ heads}) = \frac{15}{64}}$$

(b) (5 points) Compute the probability that head appears both at the second and third trials.

let us fix the two spots $\underline{2}\ \underline{H}\ \underline{H}\ \underline{2}\ \underline{2}\ \underline{2}$

and vary the rest of the 4 spots

$$P(\text{Head at second \& third}) = \frac{2^4}{2^6} = \frac{1}{4}$$

$$\boxed{P(\text{head at } 2^{nd} \text{ \& third}) = \frac{1}{4}}$$

5 2-b 5 / 5

- ✓ + **5 pts** Correct
- + **2 pts** Correct total numbers 2^6
- - **1 pts** Minor mistake

3. (10 points) There are 6 people A, B, C, D, E, F. How many ways can 6 people be seated in a row so that persons A and B can sit next to each other?

let A and B be one pair, then we have "5" "people"

5! ways = 5 x 4 x 3 x 2 x 1 = 120 ways

but the group of 2 can also be AB or BA so

we do 5! x 2!

6 people can be seated 240 ways so A & B can sit next to each other

6.3 10 / 10

- ✓ **- 0 pts** Correct
- **- 4 pts** order of A and B not counted
- **- 7 pts** no explanation

4. (15 points) Assume that 50% of emails are spam emails. There is a software detecting spam emails. The probability that a spam email detected as spam is 99%, and the probability that a non-spam email detected as non-spam is 95%. If an email is detected as spam, then what is the probability that it is in fact a spam email?

let event an email is spam = A
" is nonspam = A'

let event an email is detected as spam = B
" as nonspam = B'

$P(A) = 0.5$
$P(A') = 0.5$
$P(B \mid A) = 0.99$
$P(B' \mid A') = 0.95$

$P(B' \mid A) = 0.01$
$P(B \mid A') = 0.05$

$$P(A \mid B) = \frac{P(A) \cdot P(B \mid A)}{P(A) \cdot P(B \mid A) + P(A')P(B \mid A')}$$

$$= \frac{0.5 \cdot 0.99}{0.5 \cdot 0.99 + 0.5 \cdot 0.05}$$

$$= \frac{0.495}{0.495 + 0.025}$$

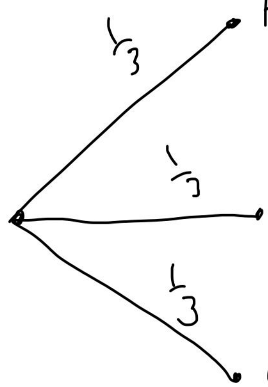
$$P(A \mid B) \approx 0.952$$

If an email is detected as spam then the probability it is spam ≈ 0.952

7.4 15 / 15

- ✓ **+ 15 pts** Correct
- **+ 10 pts** Applied Bayes theorem
- **- 3 pts** Minor mistake

5. (20 points) There are three types of coins A, B, C: coin A tosses head with probability p_1, coin B tosses head with probability p_2, coin C tosses head with probability p_3. The experimenter selects one of the three coins at random (with probability 1/3 each), and then tosses it independently 6 times. What is the probability that the experimenter see 4 many heads and 2 many tails?



$A \to P(\text{4 heads 2 Tails}) = \binom{6}{4} \cdot (p_1)^4 \cdot (1-p_1)^2$

$B \to P(\text{4 heads 2 Tails}) = \binom{6}{4} (p_2)^4 \cdot (1-p_2)^2$

$C \to P(\text{4 heads 2 Tails}) = \binom{6}{4} (p_3)^4 \cdot (1-p_3)^2$

$$P(\text{4 heads 2 tails}) = \frac{1}{3} A + \frac{1}{3} B + \frac{1}{3} C$$

$$= \frac{1}{3}(A + B + C)$$

$$= \frac{1}{3}\left(15\left(p_1^4 \cdot (1-p_1)^2 + p_2^4 (1-p_2)^2 + p_3^4 (1-p_3)^2\right)\right)$$

Probability the experimenter sees 4 heads and 2 tails $= 5\left(p_1^4 \cdot (1-p_1)^2 + p_2^4 (1-p_2)^2 + p_3^4 (1-p_3)^2\right)$

8 5 **20 / 20**

- ✓ **+ 20 pts** **Correct**
- **+ 10 pts** binomial is correct.
- **+ 3 pts** partition of events are correct
- **+ 3 pts** conditional probabilities are correct
- **+ 4 pts** final answer is correct
- **+ 6 pts** Answer is correct but conditional probabilities not explained properly
- **+ 5 pts** partial credit for binomial
- **+ 7 pts** binomial missing/wrong coefficient
- **+ 0 pts** wrong
- **+ 2 pts** incorrect binomial idea
- **+ 3 pts** regrade

6. (30 points) There are two types of six-sided dice A and B: a die A has two faces numbered 0 and four faces numbered 1, and a die B has two faces numbered -1 and four faces numbered 1. Dice A and B are rolled independently, and let X and Y be the respective outcomes of the roll.

(a) (10 points) Find the probability mass function of the random variable $W = (X - Y)^2$.

(b) (5 points) Compute the expectation W.

(c) (5 points) Compute the variance of W.

d ~~(c)~~ (5 points) Compute the r-th moments of W, $E(W^r)$.

e ~~(d)~~ (5 points) Compute the moment generating function of W.

$X \begin{cases} 0, & P(0) = \frac{2}{6} = \frac{1}{3} \\ 1, & P(1) = \frac{4}{6} = \frac{2}{3} \end{cases}$ $\quad Y \begin{cases} -1, & P(-1) = \frac{2}{6} = \frac{1}{3} \\ 1, & P(1) = \frac{4}{6} = \frac{2}{3} \end{cases}$

a) pmf of W, $W = (X-Y)^2$

$P(W = 0) = \frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9}$

$P(W = 1) = \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{2}{3} = \frac{3}{9}$

$P(W = 4) = \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}$

b) $E(W) = \sum w \cdot f(w)$, $w = 0, 1, 4$

$= (0) f(0) + (1) f(1) + (4) f(4)$

$= 0 + \frac{3}{9} + \frac{4 \cdot 2}{9} = \frac{11}{9}$

$E(W) = \frac{11}{9} = \mu$

9 6-a **10 / 10**

- ✓ **+ 10 pts** **Correct**
- **+ 5 pts** Partially correct
- **- 1 pts** Minor mistake

6. (30 points) There are two types of six-sided dice A and B: a die A has two faces numbered 0 and four faces numbered 1, and a die B has two faces numbered -1 and four faces numbered 1. Dice A and B are rolled independently, and let X and Y be the respective outcomes of the roll.

(a) (10 points) Find the probability mass function of the random variable $W = (X - Y)^2$.

(b) (5 points) Compute the expectation W.

(c) (5 points) Compute the variance of W.

d ~~(c)~~ (5 points) Compute the r-th moments of W, $E(W^r)$.

e ~~(d)~~ (5 points) Compute the moment generating function of W.

$$X \begin{cases} 0, & P(0) = \frac{2}{6} = \frac{1}{3} \\ 1, & P(1) = \frac{4}{6} = \frac{2}{3} \end{cases} \qquad Y \begin{cases} -1, & P(-1) = \frac{2}{6} = \frac{1}{3} \\ 1, & P(1) = \frac{4}{6} = \frac{2}{3} \end{cases}$$

a) pmf of W, $W = (X-Y)^2$

$$P(W=0) = \frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9}$$

$$P(W=1) = \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{2}{3} = \frac{3}{9}$$

$$P(W=4) = \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}$$

b) $E(W) = \sum w \cdot f(w)$, $w = 0, 1, 4$

$$= (0)f(0) + (1)f(1) + (4)f(4)$$

$$= 0 + \frac{3}{9} + \frac{4 \cdot 2}{9} = \frac{11}{9}$$

$$E(W) = \frac{11}{9} = \mu$$

10 6-b **5 / 5**

- ✓ **+ 5 pts** Correct
- **+ 3 pts** Partially correct
- **- 1 pts** Minor mistake

c) Given that $E(w) = \frac{11}{9} = \mu$

and variance $= E(w^2) - \mu^2 = \sigma^2$ → derived from $E[(w-\mu)^2]$

$E(w^2) = \sum w^2 f(w)$

$= (0)^2 f(0) + (1)^2 f(1) + 4^2 \cdot f(4)$

$= 0 + \frac{3}{9} + \frac{16 \cdot 2}{9}$

$E(w^2) = \frac{35}{9}$

$\sigma^2 = \frac{35}{9} - \left(\frac{11}{9}\right)^2$

$\sigma^2 = \frac{194}{81}$

Variance $\sigma^2 = \frac{194}{81}$

d) $E(w^r) = \sum u(w) \cdot f(w)$, $w = 0, 1, 4$

$u(w) = w^r$

$= (0^r) f(0) + (1^r) \cdot f(1) + (4^r) f(4)$

$= 0 + 1^r \cdot \frac{3}{9} + 4^r \cdot \frac{2}{9}$

$E(w^r) = 1^r + \frac{3}{9} + 4^r \cdot \frac{2}{9}$ r^{th} moment of W ↗

e) $E(e^{tw}) = \sum u(w) f(w) = u(0) \cdot f(0) + u(1) \cdot f(1) + u(4) f(4)$

$u(w) = e^{tw}$

$= e^{t \cdot 0} \cdot \frac{4}{9} + e^{t \cdot 1} \cdot \frac{3}{9} + e^{t \cdot 4} \cdot \frac{2}{9}$

Thus the m.g.f $= M(t) = \frac{4}{9} + \frac{3e^t}{9} + \frac{2e^{4t}}{9}$

11 6-c **5 / 5**

✓ **+ 5 pts** Correct

+ 3 pts Partially correct

- 1 pts Minor mistake

c) Given that $E(w) = \frac{11}{9} = \mu$

and variance $= E(w^2) - \mu^2 = \sigma^2 \rightarrow$ derived from $E[(w-\mu)^2]$

$$E(w^2) = \sum w^2 f(w)$$

$$= (0)^2 f(0) + (1)^2 f(1) + 4^2 \cdot f(4)$$

$$= 0 + \frac{3}{9} + \frac{16 \cdot 2}{9}$$

$$E(w^2) = \frac{35}{9}$$

$$\sigma^2 = \frac{35}{9} - \left(\frac{11}{9}\right)^2$$

$$\sigma^2 = \frac{194}{81}$$

Variance $\sigma^2 = \frac{194}{81}$

d) $E(w^r) = \sum u(w) \cdot f(w)$, $w = 0, 1, 4$

$u(w) = w^r$

$$= (0^r) f(0) + (1^r) \cdot f(1) + (4^r) f(4)$$

$$= 0 + 1^r \cdot \frac{3}{9} + 4^r \cdot \frac{2}{9}$$

$E(w^r) = 1^r + \frac{3}{9} + 4^r \cdot \frac{2}{9}$ r^{th} moment of W ↗

e) $E(e^{tw}) = \sum u(w) f(w) = u(0) \cdot f(0) + u(1) \cdot f(1) + u(4) f(4)$

$u(w) = e^{tw}$

$$= e^{t \cdot 0} \cdot \frac{4}{9} + e^{t \cdot 1} \cdot \frac{3}{9} + e^{t \cdot 4} \cdot \frac{2}{9}$$

Thus the m.g.f $= M(t) = \frac{4}{9} + \frac{3e^t}{9} + \frac{2e^{4t}}{9}$

12 6-d **5 / 5**

✓ **+ 5 pts** Correct

+ 3 pts Partially correct

- 1 pts Minor mistake

c) Given that $E(w) = \frac{11}{9} = \mu$

and variance $= E(w^2) - \mu^2 = \sigma^2$ → derived from $E[(w-\mu)^2]$

$$E(w^2) = \sum w^2 f(w)$$

$$= (0)^2 f(0) + (1)^2 f(1) + 4^2 \cdot f(4)$$

$$= 0 + \frac{3}{9} + \frac{16 \cdot 2}{9}$$

$$E(w^2) = \frac{35}{9}$$

$$\sigma^2 = \frac{35}{9} - \left(\frac{11}{9}\right)^2$$

$$\sigma^2 = \frac{194}{81}$$

Variance $\sigma^2 = \frac{194}{81}$

d) $E(w^r) = \sum u(w) \cdot f(w)$, $w = 0, 1, 4$

$u(w) = w^r$

$$= (0^r) f(0) + (1^r) \cdot f(1) + (4^r) f(4)$$

$$= 0 + 1^r \cdot \frac{3}{9} + 4^r \cdot \frac{2}{9}$$

$$E(w^r) = 1^r + \frac{3}{9} + 4^r \cdot \frac{2}{9}$$

r^{th} moment of W ↗

e) $E(e^{tw}) = \sum u(w) f(w) = u(0) \cdot f(0) + u(1) \cdot f(1) + u(4) f(4)$

$u(w) = e^{tw}$

$$= e^{t \cdot 0} \cdot \frac{4}{9} + e^{t \cdot 1} \cdot \frac{3}{9} + e^{t \cdot 4} \cdot \frac{2}{9}$$

Thus the m.g.f $= M(t) = \frac{4}{9} + \frac{3e^t}{9} + \frac{2e^{4t}}{9}$

13 6-e **5 / 5**

- ✓ **+ 5 pts** Correct
- **+ 3 pts** Partially correct
- **- 1 pts** Minor mistake

Midterm 1, 170E, Fall 2020
Instructor: Kyeongsik Nam

Printed name: TAYKHOOM DALAL

Signed name: Taykhoom Dalal

Student ID number: 305-303-123

Instructions

- Read problems very carefully. If you have any questions, send an email to the instructor.
- This is an open book, open notes, and open internet take-home exam.
- You have 24 hours to complete the exam between **Oct 26, Monday, 8AM - Oct 27, Tuesday, 8AM**. Submit the exam through the Gradescope.
- Justify everything you write as much as possible. There will be no partial credit for just guessing the correct final answer alone. Unless otherwise stated, directly citing past home-work problems or results in the lecture note and not showing your work will only get partial credit. Your solution should be mostly self-contained.

Question	Points	Score
1	15	
2	10	
3	10	
4	15	
5	20	
6	30	
Total	100	

Please sign the following statement below, and print and sign your full name afterwords.

"I assert, on my honor, that I have not received assistance of any kind from any other person and that I have not used any non-permitted materials or technologies during the period of this evaluation."

Statement:

I assert, on my honor, that I have not recieved assitance of any kind from any other person and that I have not used any non-pemitted materials or technologies during the period of this evaluation

Print your full name:

TAYKHOOM DALAL

Signature:

Ta[illegible] Dalal

14 honesty statement 0 / 0

✓ + 0 pts Correct