Math 170E: Mini Project 1

While working through this project, you may use your textbook and notes from class, a simple calculator and programming software for coding, but you are to complete the project independently and without discussion or help from anyone else. *S*how all your work and answer all questions in complete sentences with full explanations (not just answers!). Good luck!

I agree to abide by the UCLA Student Conduct Code and will follow the guidelines specified above. I will use no other resources, including talking to other people, other than those listed above. I agree that any deviation from those guidelines will result in an immediate failure of the course.

Printed name: ______________________________

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Introduction. There are many situations in which we own products that eventually break. Often, replacing a product is cheaper if you replace it *before* it breaks (consider the Apple Recycle program for example). And of course, a natural question would be how to strategize so that you always have a working device, but also pay the least amount of money on average. Clearly, if you replace your iPhone every two weeks that is likely "overkill" and you will spend way too much money, even considering that it will be cheaper to trade in your iPhone when you have a working one to return. On the other hand, if you always wait until your iPhone stops working, you will be paying a lot more for each replacement, since you can't trade in your old one.

So perhaps a strategy is to determine some number t so that you will always replace your device after t years, unless of course it breaks before then, in which case you replace it once it breaks. What is the optimal t?

Don't worry, you'll be working through this in this project!

The Project.

(1) What is your favorite kind of electronic device that you own and regularly have to replace (i.e. get a new one)? Be creative. Let us call this device "your device."

(2) In many cases, replacing your device while it is in good running condition is cheaper than replacing it after it has broken down. Suppose for the sake of this problem, that replacing your device before it has broken down costs you \$600 but replacing it after it has broken down costs you \$1500. You recall fondly your Math 170E class at

UCLA, and decide to adopt a strategy for device replacement that will minimize your long term average cost. Let us use a model that suggests the lifespan of your device is uniformly distributed between 0 and 3 years (and assume all devices are independent).

(a) What is the PDF of the lifespan of the device?

(b) What is the expected lifespan of the device?

(c) If you always wait to replace your device until it has broken down, how long do you expect (in years) it will take until you've spent \$6000? (You can include the initial cost at time zero here and assume you had to pay full price.)

(d) Suppose your initially purchased device has last so far 1 year. How long do you expect (in years) it will take until you've spent \$6000?

(3) Let's suppose you simply choose to always replace your device when it breaks down. You wonder how much money you expect to spend on average. Let's first ask the question, how much do you expect to spend in the first three years? (Note that we also count the cost of buying the first device.) Let U_1, denote the length of time the first device lasts, U_2 the length of time the second device lasts, and so on. By the assumption above, each U_i is independent and uniformly distributed on (0,3). Let N denote the number of times you have to replace the device in the first 3 years (including the initial purchase at time zero).

(a) Explain why N is itself a random variable (where does the randomness come from?).

(b) Explain why

$$N = \min\left\{n : \sum_{i=1}^{n} U_i > 3\right\}$$

In words, this says that N is equal to the smallest n such that $\sum_{i=1}^{n} U_i >$. Explain why this is.

(4) Create a script (in R or Matlab or your preferred language) that generates a sequence of such uniformly distributed random variables U_i and computes how many of them it needs in order that their sum is greater than 3 (i.e. computes N). Then adjust your script so that it runs this experiment repeatedly, say at least 50,000 times (try even more!), and takes the average of all the found N. This should approximate $\mathbb{E}N$.

(a) Explain why this should approximate $\mathbb{E}N$.

(b) What does your code suggest $\mathbb{E}N$ is?

(c) Given this, how much money would you expect to spend in the first 3 years with this strategy?

(d) Given this, is it true that if you were to carry out this strategy forever, that on average you would pay $\$1500 \cdot \mathbb{E}N/3$ per year? Why or why not?

(e) Include a printout of your code and its output with this assignment. Clearly label your code to correspond it with this question.

(5) Now let's try to verify your answers to the above, mathematically. First, let's generalize the definition of N. Consider some number $0 < c < 3$. Let N_c be the number of times you have to replace the device in the first c years. That is,

$$N_c = \min \left\{ n : \sum_{i=1}^{n} U_i > c \right\}.$$

Note that N from above is the same as N_3 in this context.

(a) Explain why $N_0 = 1$ (and thus $\mathbb{E}N_0 = 1$).

(b) Explain why, if $U_1 = u$ for some $u > c$ then $N_c = 1$.

(c) Explain why, if $U_1 = u$ for some $u < c$, then N_c has the same distribution as $1 + N_{c-u}$.

(d) For $0 < c < 3$, what is the probability that $U_1 < c$?

(e) Again assume $0 < c < 3$. Conditioned on $U_1 < c$, what is the PDF of U_1?

(f) Recall the Law of Total Expectation. This law holds for expectation that is conditioned on an event as well. For instance, suppose N and U are two random variables and A is some event (say an interval). The expected value of N conditioned on the event $U \in A$ is equal to $\int_{u \in A} \mathbb{E}(N \,|\, U = u) f(u) du$, where $f(u)$ is the PDF of U conditioned on the event $U \in A$. Use this along with your answer to (e) to show that:

$$\mathbb{E}(N_c \,|\, U_1 < c) = \int_0^c \frac{1}{c} \mathbb{E}(N_c \,|\, U_1 = u) du.$$

(g) Use parts (c) and (f) to now show that

$$\mathbb{E}(N_c \,|\, U_1 < c) = 1 + \frac{1}{c} \int_0^c \mathbb{E}(N_{c-u}) du.$$

(h) Let us denote $E_c := \mathbb{E}N_c$. You then conclude that

$$E_c = 1 + \int_0^c \frac{1}{3} E_{c-u} du = 1 + \int_0^c \frac{1}{3} E_u du.$$

Explain why each equality in the above line is true (Hint: the first has to do with probability and the second is calculus).

(i) Take the derivative (with respect to c) of the left and the right hand sides of the above equality to show that $E'_c = \frac{1}{3}E_c$.

(j) You have thus shown that $E_0 = 1$ (from part (a) above) and $E'_c = \frac{1}{3}E_c$. Explain why this means $E_c = e^{c/3}$.

(k) Now compute the expected number of replacements you'll need to make in the first three years, namely E_3.

(l) Do your answers above match those from (4b) above?

(6) Write a new script to simulate this experiment over long periods of time. In each iteration (which corresponds to a new device), randomly generate the lifespan of that device, add that to your "time" counter, TIME, and update the total cost, COST, accordingly (in this case the cost will always be \$1500 since you only replace the device when it breaks). Run this until at least 50,000 years have gone by (i.e. until TIME > 50000).

(a) Compute the average cost per year, COST/TIME and output it. (Hint: so you don't have overflow issues, use a running average that updates your average in each iteration without having to maintain large sums.) Does this coincide with your answer to (4d) above?

(b) Include a printout of this code and its output with this assignment. Clearly label your code to correspond it with this question.

(c) This doesn't seem like the best strategy to you, so you instead consider a strategy in which you will replace the device either when it breaks down, or after 1 year of it working, whichever happens first. Adjust your code to account for this strategy.
What does your code suggest the average annual cost is now?

(d) Include a printout of this code and its output with this assignment. Clearly label your code to correspond it with this question.

(7) Finally, you feel ready to ask yourself the important question – what is the optimal strategy to minimize long term costs? You are searching for the number t ($0 \le t \le 3$) such that you will pay the least amount of money on average by adopting the strategy that you will replace the device either when it breaks down, or after t years of it working, whichever happens first. Note that if you choose never to replace your device until it breaks, you are inherently using the strategy corresponding to $t = 3$ here (as in part 6a). In part 6c) you were using the strategy $t = 1$. Given this

strategy, let us call L the length of time until you replace the (first) device.

(a) Explain why

$$L = \begin{cases} t & \text{if } U_1 > t \\ U_1 & \text{if } U_1 < t \end{cases}$$

(b) Use the Law of Total Expectation to show that $\mathbb{E}L = t - t^2/6$.

(c) Now let C denote the cost that you pay for the (first) replacement. Recall this will either be \$600 (if $U_1 > t$) or \$1500 (if $U_1 < t$). Compute $\mathbb{E}C$ as a function of t.

(d) Explain the intuition that the average long term annual cost using this strategy will be $\frac{\mathbb{E}C}{\mathbb{E}L}$. (This can be proved rigorously using theory from stochastic processes, but for now, just explain the intuition).

(e) Using this, compute the average long term annual cost using this strategy, as a function of t (i.e. just compute $\frac{\mathbb{E}C}{\mathbb{E}L}$).

(f) Since your goal is to minimize this cost, use calculus to find the value of t that minimizes this function.

(g) For this optimal value of t, what is your average annual cost?

(h) Is this less than your answer for parts (6a) and (6c) above? Should it be? How much money are you saving using this strategy compared with the naive strategies in those parts?

(i) Finally, adjust your script to implement this optimal strategy and confirm your answers for this part of the problem. Again include a printout of your code and its output with this assignment. Clearly label your code to correspond it with this question.

(8) When will you next replace the device that you used for your answer to (1) above? ☺