8

Total score: 8 points Time: 50 minutes 8 questions; 1 point each

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1. Let $(X(t))_{t\geq 0}$ be an arithmetic Brownian motion with $X(0) = -2$, drift 1, volatility 2. Find $\mathbb{E}[(X(2) + X(3))^2]$.

   A. 36
   B. 37
   C. 68
   D. 69
   E. 70

$$E(\underbrace{X(2))^2}_{8} + E(\underbrace{X(3))^2}_{13} + 2E(X(2)X(3))$$

$X(2) = -2 + 2 + 2Z(2) = 2Z(2)$

$E(X(2))^2 = 4E(Z(2))^2 \Rightarrow E(Z(2))^2 = 2.$

$X(3) = -2 + 3 + 2Z(3)$

$(X(3))^2 = 1 + 4Z(3)^2 + 4Z(3)$

$E\downarrow = 1 + 12 = 13$

$E X(2)^2 + E X(2)(X(3) - X(2))$

$8 + E X(2)\, E X(1).$ $\quad 8 \times 2 = 16$

2. Let $Z \sim N(0,1)$. Let $(S(t))_{t\geq 0}$ be a geometric Brownian motion with $S(0) = 25$, drift -1 and $\mathbb{P}(S(2) \leq 50) = \mathbb{P}(Z \leq 0.5)$. Find the volatility.

   A. $\dfrac{\sqrt{2}}{2+\ln 2}$
   B. $\dfrac{2\sqrt{2}}{2+\ln 2}$
   C. $\sqrt{2}(2+\ln 2)$
   D. $\dfrac{2+\ln 2}{\sqrt{2}}$
   E. $\dfrac{2+\ln 2}{2\sqrt{2}}$

$S(2) = S(0) \cdot e^{-2+\sigma(Z(2))}$

$= 25 \cdot e^{-2+\sigma(Z(2))}$

$e^{-2+\sigma(Z(2))} \leq 2.$

$\sigma\sqrt{2} \Leftarrow \sigma Z(2) \leq \dfrac{\ln 2 + 2}{\sigma\sqrt{2}} = \dfrac{1}{2}.$

$\sigma\sqrt{2} = \dfrac{2\ln 2 + 4}{\sqrt{2}}$

$= \sqrt{2}\ln 2 + 2\sqrt{2}.$

3. Let $(Z(t))_{t\geq 0}$ be a standard Brownian motion. Find $\mathbb{E}(e^{Z(3)}|Z(1) = -1)$.

A. 1

B. $e^{-0.5}$

C. $e^{0.5}$

D. e^{-1}

E. e

$\cancel{Z(3)}$ $Z(3) = Z(1) + (Z(3) - \cancel{Z(1)}) = -1 + (Z(3) - Z(1))$

$Ee^{-1+(Z(3)-Z(1))}$. $\sim N(0, 2)$.

$= Ee^{-1+\sqrt{2}Z}$.

$= e^{-1} \cdot Ee^{\sqrt{2}Z}$

$= e^{-1} \cdot e^{1}$.

4. Let $(X(k))_{k\geq 0}$ be a symmetric random walk. Then $\mathbb{P}(X(10) \geq X(3) + 6) =$

A. $\frac{1}{2^4}$

B. $\frac{1}{2^5}$

C. $\frac{1}{2^6}$

D. $\frac{1}{2^7}$

E. $\frac{1}{2^8}$

$X(10) - X(3) \geq 6$.

$X(7) \geq 6$.

$\frac{1}{2}(X(7) + 7) = 7$.

$\binom{7}{7} \cdot \frac{1}{2^7}$

5. Let $(X(k))_{k\geq 0}$ be a symmetric random walk. Which of the following pairs of random variables is/are independent?

I. $X(7)$ and $X(4) - X(2)$

II. $X(7) - X(4)$ and $X(4) - X(2)$

III. $X(7) - X(4) + X(2)$ and $X(4) - X(2)$

A. I and II only

B. I only

C. II and III only

D. II only

E. I and III only

$(X(7) - X(4)) + (X(2) - X(0))$ and $(X(4) - X(2))$

$\operatorname{cov}(X(7) - X(4) + X(2),\ X(4) - X(2))$

$= E(X(7) - X(4) + X(2))(X(4) - X(2))$

$= E(X(7) - X(4))(X(4) - X(2)) + E\,X(2)(X(4) - X(2)) = 0.$

6. Let $(S(k))_{k\geq 0}$ be a geometric random walk with $S(0) = 18$, $u = 2$, $d = \frac{1}{3}$, $p = \frac{2}{3}$. Find $\mathbb{E}\max(15 - S(2), 0)$.

A. 19/9

B. 22/9

C. 25/9

D. 28/9

E. 31/9

$S(0)$	$S(1)$	$S(2)$	$\max(15 - S(2), 0)$
18	$\frac{2}{3}$: 36	$\frac{2}{3}$: 72	0
		$\frac{1}{3}$: 12	3
	$\frac{1}{3}$: 6	$\frac{2}{3}$: 12	3
		$\frac{1}{3}$: 2	13

~~max(15 − S(2), 0)~~

$\frac{12}{9} + \frac{13}{9} = \frac{25}{9}$

7. Let $(X(t))_{t\geq 0}$ be a Poisson process with rate 3. Find $\mathbb{E}(X(2.5)^2|X(1.5)=2)$.

A. 22

B. 24

C. 25

D. 27

E. 28

E

$X(2.5) = 2 + (X(2.5) - X(1.5))$

$= 2 + X(1)$

$4 + X(1)^2 + 4X(1)$

$4 + E X(1)^2 + 4E X(1)$ $\sim$ Poisson(3).

$4 + 12 + 12 ..$

8. Let $(X(t))_{t\geq 0}$ be a Poisson process with rate 2. Let T_1 and T_2 be the time it takes for $X(t)$ to reach 1 and 2, respectively.

$$T_1 = \inf\{t > 0 : X(t) = 1\}.$$

$$T_2 = \inf\{t > 0 : X(t) = 2\}.$$

$X(3) \sim$ Poisson(6).

Find $\mathbb{P}(T_1 \leq 3 < T_2)$.

A. $3e^{-6}$

B. $6e^{-6}$

C. $9e^{-6}$

D. $12e^{-6}$

E. $18e^{-6}$

$P(T_1 \leq 3) = P(X(3) \geq 1)$

$= 1 - P(X(3) = 0)$

$= 1 - e^{-6}$

$P(T_2 > 3) = P(X(3) < 2)$

$= P(X(3) = 0) + P(X(3) = 1)$

$= e^{-6} + 6e^{-6} = 7e^{-6}$

$P(X(3) = 1) = e^{-6} \cdot 6$

End of exam