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1. Short-answer questions.

(a) State the Central Limit Theorem.

For large n, let $S_n = X_1 + X_2 + \cdots + X_n$, X_i are identical independently distribution, then $\frac{S_n - nE[X_1]}{\sqrt{n}\,\delta(X_1)} \sim N(0,1)$

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State formally

(b) State the Chebyshev Inequality.

✓ $P(|X - E[X]| > c) \leq \frac{Var(X)}{c^2}$, for $\forall c > 0$

(c) Define: A sequence of random variables X_1, X_2,... converges to a random variable X with probability 1.

✓ $P(\lim_{n \to \infty} X_n = X) = 1$

(d) Consider a sequence of independent random variables X_n that are uniformly distributed in the interval $[0, 1]$, and let $Y_n = \min\{X_1, ..., X_n\}$. Does Y_n converge in probability? If so, what is the limit? You only need to either answer "No", or "Yes" with the identification of the limit. No explanation/computation is required.

✓ Yes, limit is 0

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2. State whether each of the following is true or false, AND supply a valid reasoning.

(a) If a sequence of random variables X_1, X_2, ... converges in probability to a constant c, then the sequence also converges to c with probability one.

False, converges to c with probability 1 requires finite many of X_i st. $X_i \neq c$. Counterexample: $P(X_i = x_i) = \begin{cases} \frac{1}{n} & \text{if } x_i = 1 \\ \frac{n-1}{n} & \text{if } x_i = 0 \\ 0 & \text{otherwise} \end{cases}$

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Then the sequence converges in probability to 0 but doesn't converges to 0 with probability 1 due to infinite 1 occurs in the sequence has non-zero probability

what exactly is the sequence? what is n?

(b) For any random variable X, we have

Yes

$$\mathbb{P}(|X| \geq a) \leq \frac{\mathbb{E}[X^4]}{a^4}, \quad a > 0. \tag{1}$$

By Chebyshev's law $P(X^4 \geq a^4) \leq \frac{E[X^4]}{a^4}$, $X^4 \geq 0$, $a > 0$

✓ if $a > 0$ $P(|X| \geq a) \leq \frac{E[X^4]}{a^4}$

(c) For a sequence of random variables X_n and a random variable X, if $\lim_{n\to\infty} \mathbb{E}[(X_n - X)^2] = 0$, then X_n converges to X in probability.

Yes

$\varepsilon > 0$, by Chebyshev's Inequ

✓ $P(|X_n - X| > \varepsilon) = P((X_n - X)^2 > \varepsilon^2) \leq \frac{E[(X_n - X)^2]}{\varepsilon^2}$

As $n \to \infty$, $\lim_{n\to\infty} P(|X_n - X| > \varepsilon) \leq \frac{0}{\varepsilon^2} = 0$

So X_n converges to X in probability

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3. Assume that the moment generating function (MGF) of a certain random variable X is

$$M_X(s) = e^{\frac{s^2}{2}+\lambda(e^s+e^{2s}-2)}, \quad \lambda > 0, \quad s \in \mathbb{R}. \tag{2}$$

Find $Var(X)$.

$$E[X] = \frac{d M_X(s)}{ds}\bigg|_{s=0} = e^{\frac{s^2}{2}+\lambda(e^s+e^{2s}-2)} \cdot \left(s+\lambda e^s+2\lambda e^{2s}\right)\bigg|_{s=0}$$

$$= e^0 \cdot (\lambda+2\lambda) = 3\lambda \checkmark$$

$$E[X^2] = \frac{d^2 M_X(s)}{ds^2}\bigg|_{s=0} = \frac{d}{ds}\frac{d M_X(s)}{ds}\bigg|_{s=0}$$

$$= \frac{d}{ds}\left(e^{\frac{s^2}{2}+\lambda(e^s+e^{2s}-2)} \cdot (s+\lambda e^s+2\lambda e^{2s})\right)\bigg|_{s=0}$$

$$= \Big(e^{\frac{s^2}{2}+\lambda(e^s+e^{2s}-2)} (s+\lambda e^s+2\lambda e^{2s})^2$$

$$+ e^{\frac{s^2}{2}+\lambda(e^s+e^{2s}-2)} (1+\lambda e^s+4\lambda e^{2s})\Big)\bigg|_{s=0}$$

$$= 1 \times (\lambda+2\lambda)^2 + 1 \times (\lambda+4\lambda)$$

$$= 9\lambda^2+5\lambda+1$$

$$Var(X) = E[X^2] - (E[X])^2 = 9\lambda^2+5\lambda - 9\lambda^2 = 5\lambda+1$$

4. Toss a fair coin independently till the first head comes up. Let the number of tosses be N. Toss another fair coin independently till the N-th head comes up. Let the number of tosses of the second coin be Y. Find the moment generating function (MGF) of Y.

$N \sim Geom(\frac{1}{2})$, Let $X_i \sim Geom(\frac{1}{2})$

$Y = X_1 + \cdots + X_N$

$E[e^{X_1 s} \cdots e^{X_N s} | N=n] = (E[e^{X_1 s}])^n = (M_{X_1}(s))^n$

$M_Y(s) = E[e^{Ys}] = E[e^{X_1 s + \cdots + X_N s}] = E[E[e^{X_1 s} e^{X_2 s} \cdots e^{X_N s} | N]]$

$= E[(M_{X_1}(s))^N] = E[e^{N \log(M_{X_1}(s))}]$

$= M_N(\log M_{X_1}(s))$

Nice! But you may just quote this existing result.

$M_{X_1}(s) = E[e^{X_1 s}] = \sum_{x=1}^{\infty} e^{X_1 s} p(1-p)^{X_1 - 1}$

$= \sum_{x=1}^{\infty} pe^s (e^s(1-p))^{X_1 - 1} = pe^s \sum_{x=0}^{\infty} (e^s(1-p))^{X_1}$

$= pe^s \frac{1}{1 - e^s(1-p)} = \frac{pe^s}{1+p-e^s}$ if $e^s(1-p) < 1$ $(1-p) \neq 0$

$s < \log(\frac{1}{1-p})$

Due to iid?

$M_N(s) = \frac{pe^s}{1+p-e^s}$, since $p = \frac{1}{2}$

(+1) extra credit

$M_Y(s) = \dfrac{\frac{1}{2} \frac{\frac{1}{2}e^s}{1+\frac{1}{2}-e^s}}{1 + \frac{1}{2} - \frac{\frac{1}{2}e^s}{1+\frac{1}{2}-e^s}} = \dfrac{\frac{1}{4}e^s}{\frac{3}{2}(\frac{3}{2}-e^s) - \frac{1}{2}e^s}$

More explanation is needed here

$= \frac{e^s}{9 - 6e^s - 2e^s} = \frac{e^s}{9 - 8e^s}$ if $s < \log 2$

(−1)

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5. Let X_1, X_2, ... be a sequence of independently distributed exponential random variables with parameter $\lambda = 2$. Does the sequence $Z_n = \dfrac{X_1 + ... + X_n}{X_2^2 + ... + X_{n+1}^2 + 1}$, $n = 1, 2, ...$, converge with probability 1? If so, what is the limit? Verify your conclusion.

$X_i \sim Exp(2)$ $\quad E[X_i] = \frac{1}{2}$ $\quad E[X_i^2] = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$$E[Z_n] = E\left[\frac{X_1 + \cdots + X_n}{1 + X_2^2 + \cdots + X_{n+1}^2}\right]$$

$$= E\left[\frac{X_1}{1 + X_2^2 \cdots X_{n+1}^2}\right] + E \cdots + E\left[\frac{X_n}{1 + \cdots X_{n+1}^2}\right]$$

$$= \frac{E[X_1]}{1 + E[X_2^2] + \cdots E[X_{n+1}^2]} + \cdots + E\left[\frac{1}{\frac{1 + \cdots + X_{n-1}^2 + X_{n+1}^2}{X_n} + X_n}\right]$$

$$< \frac{1}{n} + \cdots + \frac{1}{A + \frac{1}{2}} \; ?$$

$$\sum_{n=1}^{\infty} E[Z_n] < \infty, \text{ so } E \sum_{n=1}^{\infty} Z_n < \infty$$

So the sequence should converge

If no Expectation is infinite.

So limit is 0 ?

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