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1. State whether each of the following is true or false. No explanation is required.

(a) If X and Y are continuous random variables with PDFs $f_X(x)$, $x \in \mathbb{R}$ and $f_Y(y)$, $y \in \mathbb{R}$, then $Z = X - Y$ has the PDF $f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(x - z)dx$.

True X

(b) If X and Y are independent, then $\mathbb{E}[X^2Y^3|Y] = Y^3\mathbb{E}[X^2]$.

$Y^3 E[X^2|Y]$

True

(c) If X and Y are independent, then $Var(XY|Y) = YVar(X)$.

$Var(XY|Y) = Y^2 Var(X|Y) = Y^2 Var(X)$

False

(d) X is a random variable. Let $Z = \mathbb{E}[X|Y]$, and $W = Z - X$. Then Z and W are independent.

$cov(E[X|Y], Z - X)$

False

Counter example: $X = SY$, $S \perp Y$ (indep)

$S \sim Bern(\frac{1}{2}) \Rightarrow W \& Z$ not indep.

(e) There exists a pair of random variables X and Y, such that $\mathbb{E}[XY] = 4$, $\mathbb{E}[X^2] = 2$ and $\mathbb{E}[Y^2] = 2$.

~~True~~

False

Cauchy Schwartz

Problem

2. (a) Let R be a random variable with the PDF

$$f_R(r) = \begin{cases} re^{-r^2/2}, & r \geq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

$Y^{\frac{1}{2}}$

Find the distribution of $Y = R^2$.

$F_Y(y) = P(R^2 \le y) = P(-\sqrt{y} \le R \le \sqrt{y}) = F_R(\sqrt{y}) - F_R(-\sqrt{y})$

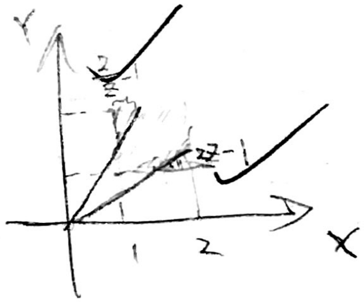
$f_Y(y) = f_R(\sqrt{y})\left(\frac{1}{2}y^{-\frac{1}{2}}\right) + f_R(-\sqrt{y})\left(\frac{1}{2}y^{-\frac{1}{2}}\right)$

$= \begin{cases} \sqrt{y}e^{-\frac{y}{2}} \times \frac{1}{2}y^{-\frac{1}{2}}, & \sqrt{y} \ge 0, \ y \ge 0 \\ 0 & \text{otherwise} \end{cases}$?

need to explain why this term vanishes

(-1)

(b) Let X and Y be independent random variables that are uniformly distributed on the interval $[1, 2]$. Find the PDF of the random variable $Z = Y/X$.



$F_Z(z) = P\left(\frac{Y}{X} \le z\right)$ $\quad Y = zX$

$= P(Y \le zX)$

$= \begin{cases} 0 & z \in (-\infty, \frac{1}{2}] \\ \frac{1}{2}\left(\frac{2(z-1)}{z}\right)^2 = 2z - 2 - \frac{1}{2}z^{-1} & z \in (\frac{1}{2}, 1] \\ 1 - \frac{1}{2}\left(\frac{2}{z} - 1\right)(2 - z) = -2z^{-1} + 3 - \frac{1}{2}z & z \in (1, 2] \\ 1 & z \in (2, \infty) \end{cases}$

(-2)
More process is needed

So $f_Z(z) = \begin{cases} 2 + \frac{1}{2}z^{-2} & z \in [\frac{1}{2}, 1) \\ 2z^{-2} - \frac{1}{2} & z \in [1, 2] \\ 0 & \text{otherwise} \end{cases}$

$Z = \frac{Y}{X}$ $\quad (X, Y) \xrightarrow{g} (W, Z)$
$W = X$ $\quad A \quad B$
$A = \{(x,y) : 1 \le x \le 2, \ 1 \le y \le 2\}$
$B = \{(w,z) \mid 1 \le w \le 2, \ \frac{1}{w} \le z \le \frac{2}{w}, \ \frac{1}{2} < z < 2\}$

3. Suppose the continuous random variables X, Y have joint density

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$$f(x,y) = \begin{cases} \dfrac{e^{-x/y}e^{-y}}{y}, & x > 0, y > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

Find $\mathbb{E}\left[e^{-X}|Y\right]$.

$E[e^{-X}|Y=y]$

$= \int_0^\infty e^{-x} f_{X|Y}(x|y)\,dx$

$f_{X|Y}(x|y) = \dfrac{\dfrac{e^{-\frac{x}{y}}e^{-y}}{y}}{\int_0^\infty e^{-\frac{x}{y}}e^{-y}y^{-1}\,dx} = \dfrac{e^{-\frac{x}{y}}e^{-y}y^{-1}}{\left[-e^{-\frac{x}{y}}e^{-y}\right]_0^\infty}$

$\left[-e^{-\frac{x}{y}}e^{-y}\right]_0^\infty = e^{-y}$, $y>0$

$= e^{-\frac{x}{y}}y^{-1}$

So $E[e^{-X}|Y=y] = \int_0^\infty e^{(-1-\frac{1}{y})x}\,y^{-1}dx$

$f_Y(y)>0$ when $y>0$

$= \left[\frac{1}{-y-1}e^{(\frac{-y-1}{y})x}\right]_0^\infty$

$= -\frac{1}{-y-1}$ $y>0$

So $E[e^{-X}|Y] = \frac{1}{Y+1}$

4. A person writes up k letters to be sent to k distinct addresses. He also gets k envelops ready with the addresses. Then he puts one letter into one envelop chosen at random without checking if they are a pair, and set them aside. Then he puts another letter into an envelope chosen at random, set aside, and keeps doing this. Eventually he gets all the letters into the envelops (one envelop contains exactly one letter). Let X be the number of pairs matched correctly in the end. Compute the variance of X. Hint: Don't try to necessarily compute the PMF of X.

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X_i indicator variable

$$Cov(X_1 + \cdots + X_k, X_1 + \cdots + X_k)$$

$$= \sum_{i=1}^{k} Var(X_i) + \sum_{i \neq j} Cov(X_i, X_j)$$

$$Var(X_i) = \frac{1}{k} \times$$

5. During the day there are X buses independently arriving at the UCLA station, where X has a Poisson distribution with parameter $\lambda = 1/2$. For each bus one of following happens with equal probability: the bus is empty, or it carries exactly 1 passenger, or it carries exactly 2 passengers. Let Y be the total number of passengers arriving at the UCLA station at the end of the day. Find $\mathbb{E}[Y]$ and $Var(Y)$.

$X \sim Po.(\frac{1}{2})$

$Y = N_1 + \cdots + N_X$

$E[Y] = E[X]E[N]$

$= 1 \times (\frac{1}{2}) = \frac{1}{2}$

please show the process.

$Var(Y) = Var(E[Y|X]) + E[Var(Y|X)]$

$= Var(X) + E[\frac{2}{3}X]$

$= \frac{1}{2} + \frac{1}{2} \cdot \frac{2}{3}$

$Var(N_1 + \cdots + N_X | X = x)$

$= Var(N_1 + \cdots + N_x)$

$= x \, Var(N_1)$

$= x \cdot \frac{2}{3}$

this is not 1

$= 1$

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