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Math 170A-5 Yeliussizov. Midterm 2

Exam time: 4:00-4:50 PM, Feb 28, 2018

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$\text{Var}\left(\frac{1-p}{p^2}\right)$

There are 5 problems.
No books, notes, calculators, phones, conversations, etc.
Turn off your cell phones.

$\text{Exp}\ \frac{1}{p}$

$(1-p)^{k-1}p$

P 1 (8)	P 2 (8)	P 3 (8)	P 4 (8)	P 5 (8)	Total (40 pt)
8	8	7	4	3	30

$\sum_0^\infty a^t = \frac{1}{1-a}$ $\sum_0^\infty t a^t = \frac{a}{(1-a}$

	bernulli	binomia	geometric	poisson
Px(k)	$\begin{cases} p \\ 1-p \end{cases}$	$\begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & 0 \ldots n \\ 0 & \text{else} \end{cases}$	$\begin{cases} (1-p)^{k-1} p & 1 \ldots \infty \end{cases}$	$\begin{cases} \frac{e^{-\lambda}\lambda^k}{k!} & k = 0, 1, 2 \ldots n \end{cases}$
Exp(X)	p	np	$\frac{1}{p}$	λ
Var(X)	$p(1-p)$	$np(1-p$	$\frac{(1-p)}{p^2}$	

$\frac{a(1-r^{n+1})}{1-r}$

Problem 1. Let X be a geometric random variable with parameter $p = 1/2$. Let $Y = \max(X, 3)$.
(a) (4 points) Find the PMF of Y.
(b) (4 points) Find the expectation of Y.

a) $P_X(k) = \begin{cases} \left(\frac{1}{2}\right)^k & k=1,2,3\ldots \\ 0 & \text{else} \end{cases}$

$\max(4,3) = 4$

$\max(1,3) = 3$ $\max(2,3) = 3$

$\max(3,3) = 3$

$P_Y(y) = \begin{cases} \frac{7}{8} & \text{if } y = 3 \\ \left(\frac{1}{2}\right)^y & \text{if } y = 4,5,6\ldots \\ 0 & \text{else} \end{cases}$

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$P_Y(y=3) = P_X(k=1) + P_X(k=2) + P_X(k=3)$

$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$

$= \frac{4}{8} + \frac{2}{8} + \frac{1}{8} = \frac{7}{8}$

b) $E[Y] = \sum_y y P_Y(y) = \sum_x x P_X(x) + 1 + \frac{1}{4}$

$\frac{8}{8} + \frac{2}{8}$

as for $\sum_1^3 x P_X = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} = \frac{4}{8} + \frac{4}{8} + \frac{3}{8} = \frac{11}{8}$

$3 P_Y(3) = \frac{21}{8}$

and $\sum_{y=4}^{\infty} y P_Y(y) = \sum_{x=4}^{\infty} x P_X(x)$

$\therefore \sum_y y P_Y(y) = E[X] + \frac{5}{4}$

$X \sim \text{geom}\left(\frac{1}{2}\right)$

$E[X] = \frac{1}{\frac{1}{2}} = 2$

4

$E[Y] = 2 + \frac{5}{4} = \frac{13}{4}$

Problem 2. Toss a fair coin until you see both possible outcomes (heads and tails, regardless of their order). Let X be the number of tosses made.
(a) (4 points) Find $E[X]$
(b) (4 points) Find $Var[X]$

a) let X_1 = # of tosses until 1st unique outcome
X_2 = # of tosses until 2nd unique outcome after previous unique outcome (1st unique outcome)

$E[X] = E[X_1] + E[X_2]$

$E[X_1] = 1$ as $P_{X_1}(k) = \begin{cases} 1 & \text{if } k=1 \\ 0 & \text{else} \end{cases}$

$E[X_2] = \frac{1}{\frac{1}{2}} = 2$ $\quad P_{X_2}(k) = \begin{cases} (\frac{1}{2})^{k-1}\frac{1}{2} & \text{if } k = 1,2,3\ldots \\ 0 & \text{else} \end{cases}$

$X_2 \sim geom(\frac{1}{2})$

$E[X] = E[X_1] + E[X_2]$
$= 1 + 2$
$= 3$

4

b) $Var[X] = E[X^2] - E[X]^2$

but bc X_1 and X_2 independent and $X = X_1 + X_2$

$Var[X] = Var[X_1 + X_2] = Var[X_1] + Var[X_2]$

$Var[X_1] = E[X_1^2] - E[X_1]^2$
$= 1 - 1 = 0$

$X_2 \sim geom(\frac{1}{2})$ $\quad Var[X_2] = \frac{1-p}{p^2} = \frac{1-\frac{1}{2}}{(\frac{1}{2})^2} = \frac{\frac{1}{2}}{\frac{1}{4}} = 2$

4

$Var(X) = 0 + 2 = 2$

Problem 3. In a class of n students, every student gets an A with probability p and a B with probability q (where $p+q<1$), independently of any other student. Let X and Y be the numbers of students that get an A and a B.

(a) (4 points) Find the joint PMF of X and Y.

(b) (4 points) Find the conditional PMF of X given $Y=k$.

a)

$$P_{XY}(x,y) = \begin{cases} \underbrace{\binom{n}{x}p^x}_{\text{choose my A students}}\underbrace{\binom{n-x}{y}q^y}_{\text{choose B students}}\underbrace{(1-p-q)^{n-x-y}}_{\text{rest}} & \text{if } 0\le x,y \text{ and } x+y\le n \\ 0 & \text{else} \end{cases}$$

4

$y \backslash x$	0	1	2
0	$(1-p-q)^n$	$\binom{n}{1}p(1-p-q)^{n-1}$	
1	$\binom{n}{1}q(1-p-q)^{n-1}$	$\binom{n}{1}\binom{n-1}{1}$	
2			

b)

$$P_{X|Y=k}(x|y=k) = \frac{P_{XY}(x,k)}{P_Y(k)}$$

$$P_{XY}(x,y=k) = \binom{n}{x}p^x\binom{n-x}{k}q^k(1-p-q)^{n-x-k}$$

$$P_Y(k) = \sum_x P_{XY}(x,y=k)$$

$$= \sum_{x=0}^{n-k}\binom{n}{x}p^x\binom{n-x}{k}q^k(1-p-q)^{n-x-k}$$

$$P_{X|Y}(x|y=k) = \frac{\binom{n}{x}p^x\binom{n-x}{k}q^k(1-p-q)^{n-x-k}}{\cancel{\sum_{x=0}^{n-k}\binom{n}{x}p^x\binom{n-x}{k}q^k(1-p-q)^{n-x-k}}}$$

3

Problem 4. (8 points) You are taking a test that might consist of 1 up to n questions, with all possible numbers in $\{1, \ldots, n\}$ occurring with (equal) probability $1/n$. You answer each question correctly with probability $1/2$ (independently of any other answer). Find the expected number of correct answers that you give on the test.

X = # of Q's $\quad P_X(k) \begin{cases} \binom{n}{k}\left(\frac{1}{n}\right)^k\left(1-\frac{1}{n}\right)^{n-k} & \text{if } k = 0, 1, 2, \ldots n \\ 0 & \text{else} \end{cases}$

$X \sim$ binomial $(n = n, p = \frac{1}{n})$

Y = # answered correct $\quad E[X] = n \cdot \frac{1}{n} = 1$

$E[Y] = E[Y \mid X=0]p_X(0) + E[Y \mid X=1]p_X(1) \ldots E[Y \mid X=n]p_X(n)$

$Y \mid X=k$ binomial $n = k$ $p = \frac{1}{2}$

$E[Y \mid X = k] = \frac{1}{2} \cdot k$

$$E[Y] = \sum_{k=0}^{n} \frac{1}{2} k \binom{n}{k}\left(\frac{1}{n}\right)^k\left(1-\frac{1}{n}\right)^{n-k}$$

$$= \frac{1}{2} \underbrace{\sum_{k=0}^{n} k \binom{n}{k}\left(\frac{1}{n}\right)^k\left(1-\frac{1}{n}\right)^{n-k}}_{E[X]}$$

$$= \frac{1}{2} \cdot E[X]$$

$$= \frac{1}{2}$$

Problem 5. (8 points) Let X and Y be independent Poisson random variables. Prove that $Z = X + Y$ is also a Poisson random variable.

$X \overset{\text{poisson}}{\sim} (\lambda_1)$ $\quad Y \overset{\text{poisson}}{\sim} (\lambda_2)$ $\qquad Z = X + Y$

$$P_X(x) = \begin{cases} \dfrac{e^{-\lambda_1}\lambda_1^{x}}{x!} & \text{if } x = 0,1,2,3\ldots \\ 0 & \text{else} \end{cases}$$

since poisson
$E[X] = \lambda_1$
$E[Y] = \lambda_2$

$$P_Y(y) = \begin{cases} \dfrac{e^{-\lambda_2}\lambda_2^{y}}{y!} & \text{if } y = 0,1,2,3\ldots \\ 0 & \text{else} \end{cases}$$

$E[X+Y] = E[X] + E[$

$$E[Z] = E[X+Y] = E[X] + E[Y] = \lambda_1 + \lambda_2$$

$$\mathrm{Var}[Z] = \mathrm{Var}(X+Y) \overset{\text{since independent}}{=} \mathrm{Var}(X) + \mathrm{Var}(Y) = \lambda_1 + \lambda_2$$

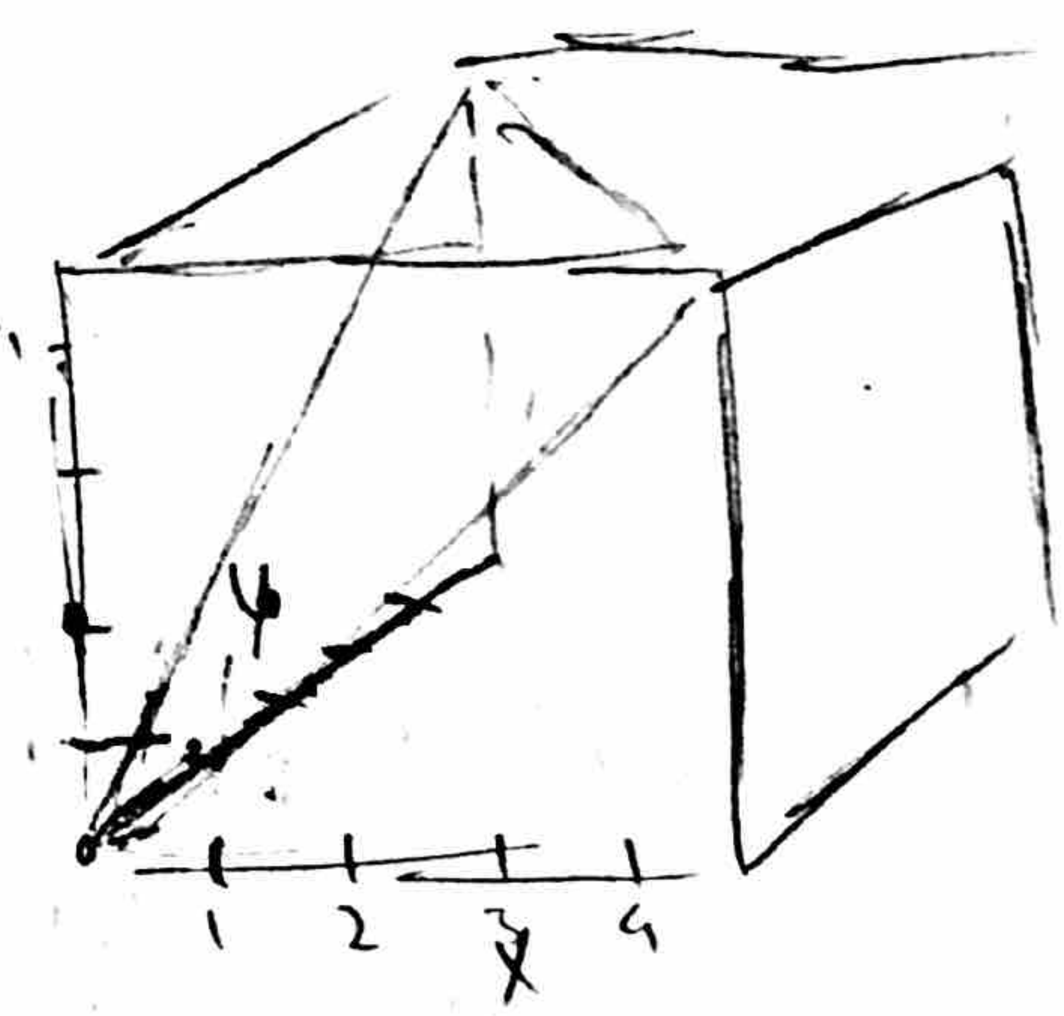
$$P_{XY}(x,y) = P_X P_Y$$

$$P_Z = \begin{cases} \sum_{i=0}^{k} \binom{k}{i} P_X(x)^{i} P_Y(y)^{k-i} \\ 0 \text{ else} \end{cases}$$

0 1
1 0
1 1
0 2
2 0
1 2
2 1
3 0
0 3
1 3
3 1
2 2
0 4
4 0

0 5
5 0
2 3
3 2
1 4
4 1

0 6
6 0
3 3



$\Downarrow$

$\therefore$ Z in Poisson$(\lambda_1 + \lambda_2)$